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Time series difference approach for evaluating sensitivity of nonlinear dynamic systems

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Abstract

This research is aimed to establish a novel approach for assessing sensitivities of nonlinear systems to initial conditions and system parameters via an evaluation of Time Series Difference. An evaluation method is proposed for measuring the differences of two trajectories representing the solutions of nonlinear systems, in responding to different initial conditions and/or system parameters. Recurrence relations are established for numerically evaluating the time series differences. Various nonlinear responses are evaluated with the approach proposed. A typical nonlinear dynamic system the Duffing system are considered for demonstrating the application of the approach in numerically and graphically assessing the sensitivities. The approach shown effectiveness in the assessment and can a useful tool for scientists and engineers in evaluating the initial-condition and system-parameter dependent sensitivities.

Keywords: Sensitivity; Time series difference; Chaos; Quasi-periodicity; System-parameter dependent sensitivity; Initial-condition dependent sensitivity; Nonlinear behavior; Nonlinear dynamic systems; Lyapunov exponents

1. Introduction

Nonlinear responses of dynamic systems are widely existing in fields of engineering and physics [1-3]. Numerous research works has been conducted to study the nonlinear behavior and sensitivities of the nonlinear systems. Among which, Lyapunov exponent is probably the most widely used method in diagnosing the nonlinear behavior such as periodic, quasi-periodic and chaos and the sensitivities of the systems to initial conditions. The main definition of Lyapunov exponent is to quantify the influence of initial conditions on the system by measuring the exponential rate of divergence of infinitesimally close orbit of smooth dynamical system [4]. A positive maximal Lyapunov exponent indicates instability of the limit set or it can be an indication for a chaotic attractor. Due to its solid theoretical foundation, the Lyapunov exponent therefore becomes a widely accepted criterion for diagnosing nonlinear behavior and initial condition dependent sensitivity of the nonlinear systems. Various methods are in fact proposed to measure of the sensitive dependence upon initial conditions. Wolf [4] proposed a method of determining the Lyapunov exponent from a time series by monitoring the long-term growth rate of small volume elements in an attractor. Müller proposed a method to calculate Lyapunov exponents for dynamic systems with discontinuities [5]. Moreover, the algorithm for the calculation of the spectrum of Lyapunov exponent is generalized for nonlinear dynamic systems with discontinuities. All the calculations of the Lyapunov exponents are focus on the sensitivity of the initial conditions, however, for a nonlinear system, the influence of system parameters is also significant for the design and analysis of nonlinear systems. Dai [6-7] proposed the Periodicity Ratio method to diagnose the nonlinear behavior by introducing an index ranging between 0 and 1. A region diagram can be calculated by the method to show different nonlinear behaviors within a large range of system parameters [8].

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Both the P-R method and Lyapunov exponent are for diagnosing the nonlinear behavior for nonlinear systems, especially for chaotic cases which may cause inverse affect in engineering applications. One of the important characteristics for chaos is that it is sensitive to the initial conditions. Numerous research works are available in the literature studying the sensitivities of nonlinear systems [9]. Schittenkopf and Deco [10] applied ε-information flow to measure the sensitive dependence on the initial conditions. Srinivasan [11] presented the analysis of soda can showing that the slight asymmetries may result in non-periodic motion with exponentially increasing sensitivity to initial conditions. Several studies are found in the literature on sensitivities of atmosphere and meteorological systems to initial conditions [12-17].

Nevertheless, the sensitivity studies are mainly focusing on initial-condition dependent nonlinear systems of many areas. With the literature available to the authors, few studies are conducted to show the sensitivities of nonlinear systems to system parameters. In this research, a new approach utilizing the evaluation of time series difference is to be established for measuring the sensitivities of the nonlinear systems to initial conditions and to system parameters. Some useful conclusions are obtained based on numerical and graphical investigations. The approach established shows effectiveness in evaluating the sensitivities of nonlinear dynamic systems to both initial conditions and system parameters.

2. Evaluating sensitivity with Lyapunov exponent and Periodicity Ratio (P-R)

The sensitivity of initial conditions has been proven to be one of the most important characteristics of chaotic systems. Numerous methods are proposed to show the influence of initial conditions on nonlinear systems, Lyapunov exponent is perhaps the most popular for evaluating the initial-condition dependent sensitivities. Assuming the solution of a nonlinear system is $x(t)$ corresponding to a given initial condition, and the solution becomes $y(t)$ when the initial condition is slightly changed. In the definition of Lyapunov exponent, if a system is allowed to evolve from two slightly differing initial conditions, x_0 and y_0, for numerical recurrence, the solution at the end of the first time step can be expressed as:

$$x_1 = f(x_0)\ ;\ y_1 = f(y_0) \quad \text{.....................................(1)}$$

With *n*th iteration, the expression can be rewritten as:

$$x_n = f(x_{n-1});\ \ y_n = f(y_{n-1}) \quad \text{..................................... (2)}$$

The difference between can be written as:

$$\left|x_n - y_n\right| = \left|f(x_{n-1}) - f(y_{n-1})\right| = \frac{\left|f(x_{n-1}) - f(y_{n-1})\right|}{\left|x_{n-1} - y_{n-1}\right|}\left|x_{n-1} - y_{n-1}\right|$$
$$= \left|\frac{df}{dx}\right|_{x_{n-1}} \left|x_{n-1} - y_{n-1}\right| = \left|\frac{df}{dx}\right|_{x_{n-1}} \left|\frac{df}{dx}\right|_{x_{n-2}} \left|x_{n-2} - y_{n-2}\right| = \left|\prod_{n=0}^{n-1}\frac{df}{dx}\right|_{x_n} \left|x_0 - y_0\right| \quad \text{..................................... (3)}$$

Take the geometric average of the difference,

$$\lambda = \lim \frac{1}{n}\sum_{n=0}^{n-1} \ln\left|\frac{df(x_n, \varepsilon)}{dx}\right| \quad \text{..................................... (4)}$$

Where λ is defined as the Lyapunov exponent which actually gives the average rate of divergence. If λ is negative, slightly separated trajectories converge and evolution is not chaotic. If λ is positive, nearby trajectories diverge, the evolution is sensitive to initial conditions.

The P-R method [6], on the other hand, is introduced for measuring the periodically overlapping points in Poincare maps. The overlapping points can be defined and numerically evaluated by the following equation.

$$NOP = \zeta(1) + \sum_{k=2}^{n}\{\zeta(k)P[\prod_{l=1}^{k-1}(X_{kl} + \dot{X}_{kl})]\} \quad \text{..................................... (5)}$$

where

$$\zeta(k) = \{\sum_{i=k}^{n}[Q(X_{ki})Q(\dot{X}_{ki})]\} \bullet P\{\sum_{i=k}^{n}[Q(X_{ki})Q(\dot{X}_{ki})] - 1\} \quad (6)$$

in which Q and $P$are step functions.

The Periodicity Ratio can then be determined by

$$\gamma = \lim_{n\to\infty} \frac{NOP}{n} \quad (7)$$

The Periodicity Ratio (P-R) value such defined can be used as an index. In the case that$\gamma = 1$, the response of the system is perfectly periodic, therefore the system is insensitive to initial conditions. When$\gamma = 0$, the system is perfectly nonperiodic and the system is sensitive to initial conditions.

3. Sensitivities of nonlinear systems to initial conditions and system parameters

The Lyapunov exponent measures the sensitivity of a nonlinear system to initial conditions, and the P-R method measures the periodicity of a system corresponding to given initial conditions. To evaluate the sensitivities of nonlinear systems, this research is to graphically investigate the sensitivities with measurements of the differences of the two trajectories at each time step corresponding the perturbations of initial conditions and system parameters. The difference of time series is plotted based on numerical simulation for the responses of nonlinear systems considered.

To study the influence of initial conditions, the value of initial conditions is slightly changed with other conditions remaining the same.

Assuming a nonlinear system has a solution $x(t)$. For numerically determining the solution, the recurrent relation can be expressed as

$$x_{n+1} = f(x_n) \quad (8)$$

When a different initial condition is considered, the solution of the system can be assumed as $x'(t)$, and the recurrent solution is therefore

$$x'_{n+1} = g(x'_n) \quad (9)$$

The difference of the two solutions at $t = n$, therefore the difference of the two trajectories plotted with the two solutions, can be given as

$$d_n = x_{n+1} - x'_{n+1} \quad (10)$$

Such an approach with Time Series Difference Evaluation can be implemented to graphically investigate the characteristics of a nonlinear system.

3.1. Initial condition dependent sensitivities

To demonstrate the sensitivity of initial conditions, a commonly used Duffing's equation in the following form can be considered.

$$\ddot{x} + K\dot{x} + x^3 = B\cos t \quad (11)$$

With the approach of Time Series Difference Evaluation, various responses such as periodic, irregular, quasi-periodic and chaos of a nonlinear system can be analyzed and visualized.

3.1.1. Periodic cases

Periodic case for the system is found as B=1 and K=0.41. Utilizing Eq. (10) and based on the numerical solutions obtained for the system governed by Eq. (11), the difference of time series is obtained as shown in Figure 1, with slightly different (1%) initial condition.

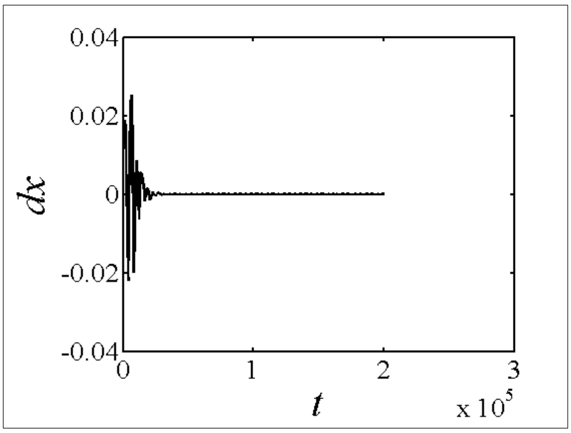

Figure 1 Difference of displacement based on change in initial conditions

From the figure, one may observe that the two trajectories of the system corresponding to two different initial conditions separate from each other for a short period of time and then become identical in terms of displacement. The initial separation of *dx* defined by Eq. (10) is due to the effects of the initial conditions to the nonlinear system.

In fact, the velocity difference of the system with two different initial conditions should become identical after the short disturbing period. This implies that the shapes of the two trajectories becomes identical after the disturbing short period, in reelecting the periodic characteristics of the system. In other words, for a system to be periodic, both the displacement difference and velocity difference of the system must be zero.

Let the first derivative of $x(t)$, or the velocity of the system be defined as

$$y = \frac{dx}{dt} = v \quad \text{.................................... (12)}$$

whereas the velocity corresponding to a slightly changed initial condition be v'.

For the recurrence relations

$$y_{n+1} = f(y_n) \quad \text{.................................... (13)}$$

the velocity recurrence relation for numerical calculations can be given as

$$dy_n = v_{n+1} - v'_{n+1} \quad \text{.................................... (14)}$$

The velocity difference for this case is shown in Figure 2.

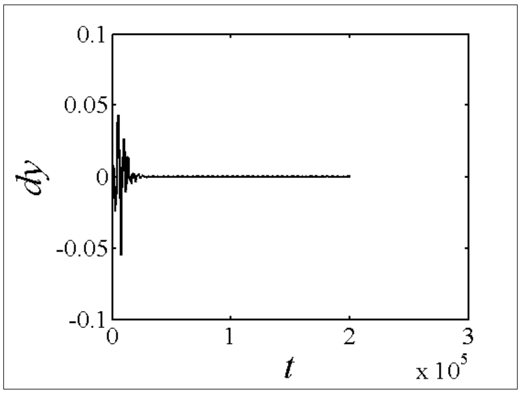

Figure 2 Difference of velocity based on change in initial conditions

With the two figures determined, one may thus conclude that this case is indeed periodic, as the displacements and shapes of the two trajectories become identical and stabilized.

3.1.2. Chaotic case

The responses of the nonlinear system governed by Eq. (11) may not necessarily be periodic, corresponding to slightly change initial conditions. A non-periodic case with B = 8 and K=0.1 is found, and the corresponding difference of time series and difference of velocity are plotted in Figures 3 and 4 respectively.

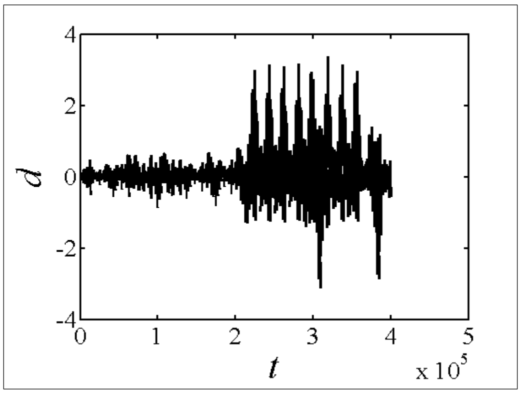

Figure 3 Difference of time series based on change in initial condition for the chaotic case

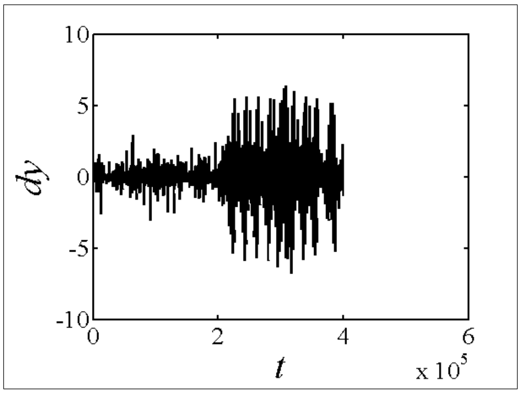

Figure 4 Difference of velocity based on change in initial condition for the chaotic case

As can be seen from the two figures, the two differences vary dramatically with significantly large amplitude in comparing with that of the periodic case. The figures imply that the displacements and shapes of the two trajectories of the solutions corresponding to two different initial conditions are significantly different in a random pattern and the differences are not reduced or stabilized with time. This implies a chaotic case.

With the Poincare map plotted for this case, as shown in Figure 5. The response of the system is indeed chaotic for this case.

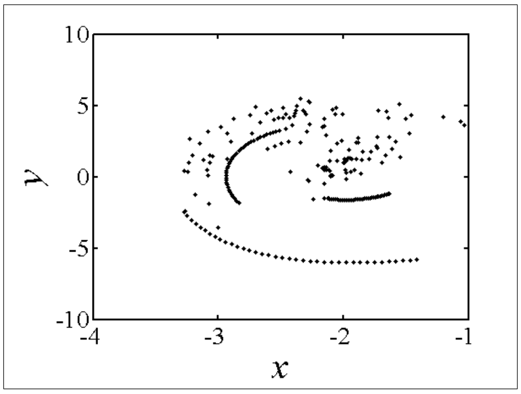

Figure 5 Poincare map of the system *B*= 8, *K*=0.1

3.1.3. Quasi-periodic case

Quasi-periodic cases are also seen in nonlinear systems. A quasi-periodic case usually shows a continuous curve or loop in Poincare map. The response in quasi-periodic case is neither periodic nor chaotic. One such case in found for the system governed by Eq. (11), as *K*=0.01 and *B*=5.25. The displacement and velocity difference curves are shown in Figures 6 and 7, corresponding to slightly difference initial conditions.

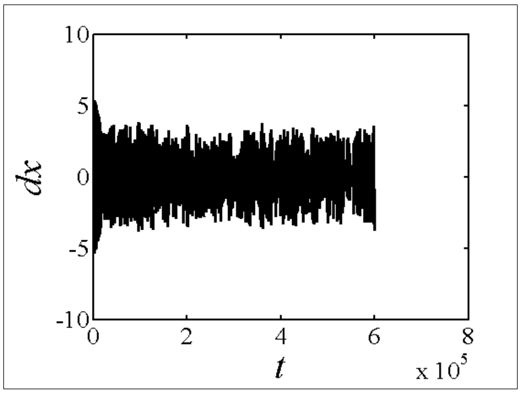

Figure 6 Difference of time series based on change in initial condition

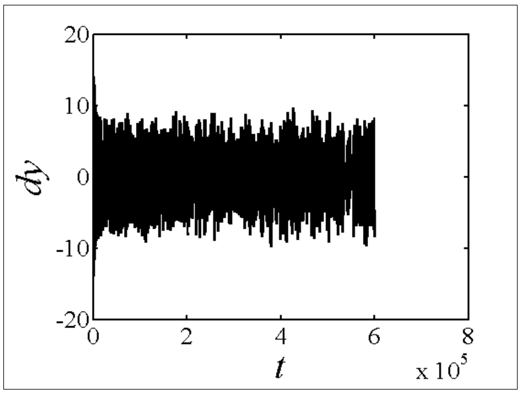

Figure 7 Difference of time series based on change in initial condition

From the Poincare map plotted for this case, as shown in Figure 8, one may observe that this case is a quasi-periodic case.

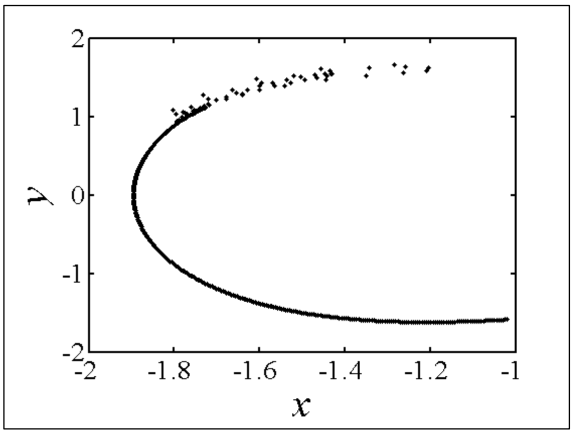

Figure 8 Poincare map of the system *B*= 5.25, *K*=0.01, for a quasi-periodic case

3.2. System parameter dependent sensitivities

Analogically, with the time series difference evaluation approach discussed above, the sensitivity of a nonlinear system to its system parameters can be evaluated. To evaluate this type of sensitivity, one may consider the response of the system to a perturbation of a system parameter. This may obviously cause difficulties. In considering the sensitivity to initial conditions, the nonlinear system itself is not changed rather than a slightly changed initial conditions. Any change on system parameter means the change of the system or the governing equation. In order to evaluate a system's dependence on the system-parameters, with the time series difference evaluation approach, one may therefore have to consider the relations of two trajectories or two solutions of two different systems. The difference of the two systems is due to slightly changed system parameters. This is the significant difference from the approach implementing Lyapunov exponent.

For many nonlinear dynamic systems in engineering practice, a nonlinear system can be described by the following equation of general form.

$$\ddot{x} + 2c\dot{x} + \omega^2 x = \Phi(t, x, \dot{x}) \quad \text{.................................. (15)}$$

Where $\Phi(t, x, \dot{x})$ can be a nonlinear function desired? Duffing's equation is a special case of the general governing equation of Eq. (15). To change any of the parameters shown in the equation means to set a new governing equation, regardless how small the parameter change is.

For the sake of clarification, the solution of the system and its first derivative are still designated as *x* (*t*) and y (*t*) = d*x*/d*t* = *v*. When a parameter of the system is slightly changed, the solution is denoted as *X* (*t*) and the velocity *V* (*t*) corresponding to the different system with changed parameter. As such, the difference of the two trajectories corresponding to the solutions *x*(*t*) and *X*(*t*) of the unchanged system and the system with changed parameter can be calculated with the following recurrence relation.

$$d_n = x_{n+1} - X_{n+1} \quad \text{.................................. (16)}$$

The difference of the two velocities of the changed and unchanged systems can then be calculated by the following recurrence relation.

$$dy_n = v_{n+1} - V_{n+1} \quad \text{.................................. (17)}$$

3.2.1. Periodic case

Consider the periodic case used for evaluating the initial-condition dependent sensitivity, with *B*=1 and *K*=0.4 for the system governed by Eq. (11). With a slightly changed parameter *K*, the displacement and velocity differences are plotted as shown in Figures 9 and 10 respectively, by implementing the recurrence relations shown in Eqs. (16) and (17).

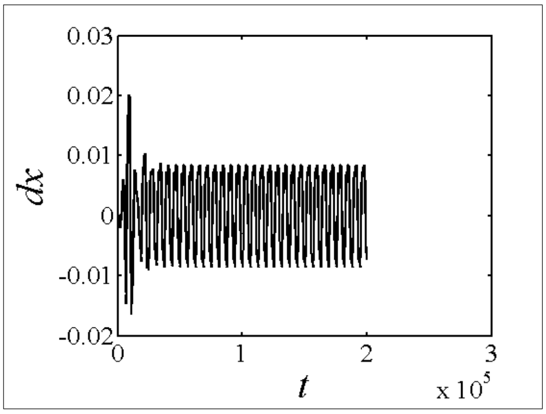

Figure 9 Difference of displacement based on a slightly changed *K*

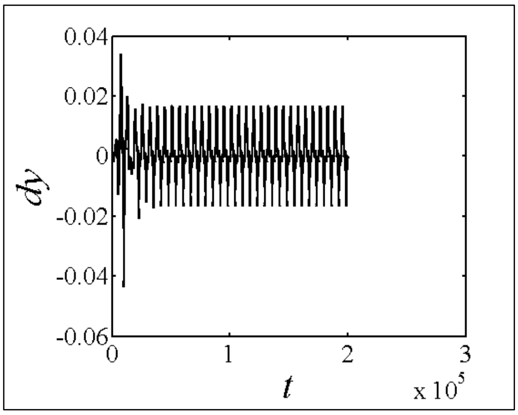

Figure 10 Difference of velocity based on a slightly changed *K*

As can be seen from the two figures above, the differences are very small around zero both for displacement difference and velocity difference. Also, the two curves show very good stabilization after the short period of disturbance, corresponding to the increase of time *t*. This implied a periodic case. In other words, the system is no sensitive to parameter *K* in this case.

In fact, for this periodic case, the system is not sensitive to parameter *B* either. This is shown in Figures 11 and 12.

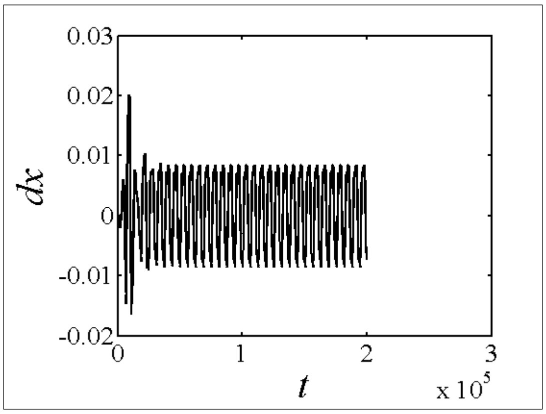

Figure 11 Difference of displacement based on change in *B*

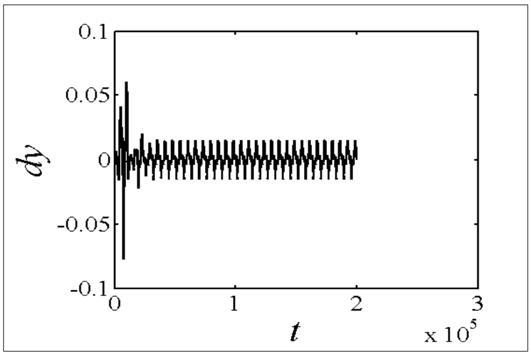

Figure 12 Difference of velocity based on change in B

For all the numerical simulations did in this research, in fact, the system considered is not sensitive to system parameters in periodic cases.

3.2.2. Chaotic case

Chaotic cases are found in the responses of the system. For these cases, the system is sensitive to system parameters. One of the chaotic cases is at B=12 and K=0.1 of the system. As can be seen from Figure 13, the displacement difference plotted as per Eq. (16) varies dramatically and continuously vary in a random fashion as time increases. This implies that a slight change in parameter *K* causes significant variation in the response of the system. Separation of the two trajectories.

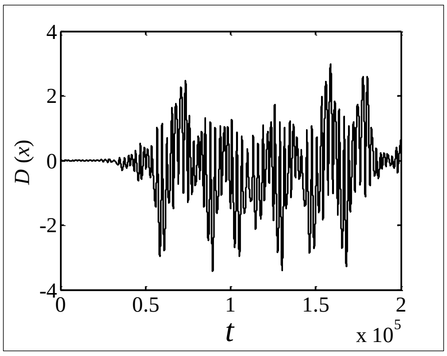

Figure 13 Difference of displacement corresponding to a slight change of parameter *K* in a chaotic case

In this case, the system is also sensitive to parameter B, as shown in Figure 14, where the differences of the trajectories vary significantly in comparing with that of periodic cases and with a significant disturbance pattern. One may therefore conclude that this is a chaotic case. Based on the numerical simulations for this research, in fact, the system considered is sensitive to system parameters in all chaotic cases.

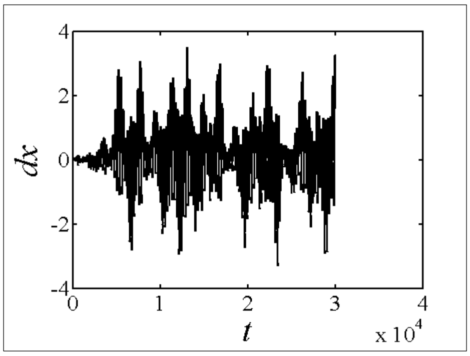

Figure 14 Difference of displacement corresponding to a slight change in parameter *B* in a chaotic case

3.2.3. Quasi-periodic case

Figures 15 and 16 show the sensitivity of the system governed by Eq. (11) to system parameters *K* and *B* respectively, for a quasi-periodic case when *B*=3, *K*=0.001. It can be seen from the figures; the displacement differences are dramatically large and the significant variation of the differences continues with time. The large differences of the displacements in comparing with that of the periodic cases are due to slight change of the parameters, for this quasi-periodic case. As per the numerical simulations of this research, the sensitivity of all the quasiperiodic cases is system-parameter dependent.

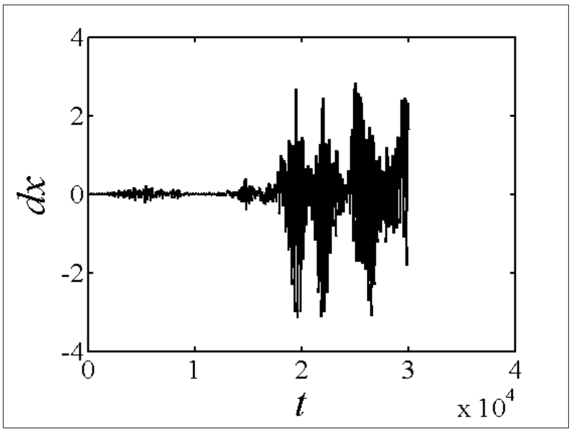

Figure 15 Displacement difference due to slight change in parameter *K* in a quasi-periodic case

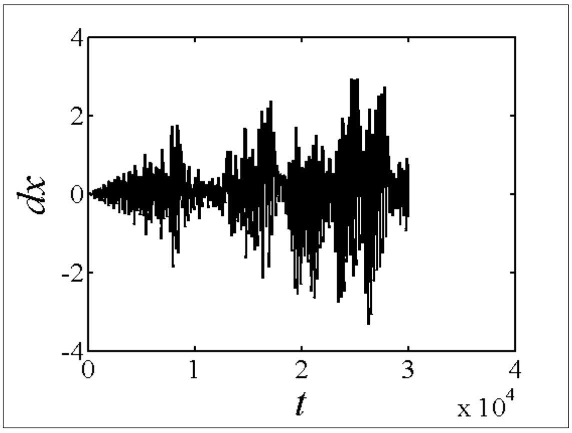

Figure 16 Displacement difference due to slight change in parameter *B* in a quasi-periodic case

Based on the discussion above, it can be stated that nonlinear systems are sensitive to system parameters, especially for chaotic and quasiperiodic cases. When a nonlinear system is periodic, the system is not sensitive to initial conditions nor to system parameters. It can also be stated that the approach of time series difference evaluation is an effective tool for analyzing and evaluating sensitivities of a nonlinear system to initial conditions and to system parameters.

4. Conclusion

Conventionally, the sensitivity analyses are mainly focusing on the sensitivity to initial conditions. This research proposes an approach of time series difference evaluation for evaluating the sensitivities of nonlinear systems to initial conditions and to system parameters. The approach proposes shows effectiveness in evaluating and analyzing these sensitivities for nonlinear systems. Based on the findings of this research, the following conclusions can be drawn.

- Chaotic system is not just sensitive to initial conditions, it is also sensitive to system parameters, if nonlinear systems are considered.
- Periodic systems are insensitive to either initial conditions or system parameters, for the nonlinear systems.

- Slight variation of system parameters may cause significant variation and randomness of a nonlinear system's response, and may cause chaos or quasi-periodicity of the system. The approach of time series difference evaluation can numerically determine and graphically illustrate the variations.

With the findings of the present research, it is significantly important for scientists and engineers to pay attention to system-parameter dependent sensitivities in analyzing and designing nonlinear systems, as slight variation of system parameters may cause unwanted responses such as chaos.

Compliance with ethical standards

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Disclosure of conflict of interest

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest, or non-financial interest in the subject matter or materials discussed in this paper.

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