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Analysis of nonlinear vibrations of two-degree-of-freedom systems

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Abstract

In this paper, the motions of two-mass systems with two-degree-of-freedom are investigated by using an analytical approach. The masses are connected by linear and nonlinear springs. The motions of systems are described by systems of two coupled strong nonlinear differential equations. Nonlinear differential equations are transferred into a single equation by using some intermediate variables. An analytical method, the equivalent linearization method, is employed to analyze the free nonlinear vibration of systems. The oscillation systems with different values of the parameters are investigated in this paper. In order to verify the accuracy of the obtained results, the present solutions are compared with those achieved by the Hamiltonian approach and the exact solutions. The comparison results show that the obtained solutions are more accurate than those obtained by the Hamiltonian approach.

Keywords: Nonlinear vibration; Two-degree-of-freedom; Equivalent linearization; Weighted averaging

1. Introduction

The motion of nonlinear two-degree-of-freedom (TDOF) oscillation system has been widely investigated in the past few decades [1, 2, 3, 4, 5, 6]. TDOF systems are important in engineering because many practical engineering components consist of coupled vibrating systems that can be modeled by using TDOF systems such as elastic beams supported by two springs and vibration of a milling machine. The equations of motion of nonlinear TDOF system consist of two second-order differential equations with cubic nonlinearities.

Due to limitation of existing exact solutions, many analytical approaches have been developed such as the harmonic balance method [7], the Hamiltonian Approach [8], the Parametrized Perturbation method [9], the Variational Iteration method [10], the Homotopy Perturbation method [11], the Energy Balance method [12 , 13] and the linearization equivalent method [14, 15]. These analytical methods are useful techniques for quantitative analysis of nonlinear oscillation systems.

The advantage of the linearization equivalent method [14, 15] is that this method is very simple and convenient to apply. The obtained results are normally accepted for oscillators with weak nonlinerity. Nevertheless, the accuracy of the Equivalent Linearization Method with conventional averaging value normally reduces for middle or strong nonlinear systems [14, 15, 16, 17]. A reason is that some terms will vanish in the averaging process, for example, the averaging value of the functions *sin(t)* and *cos(t)* over one period is equal to zero. Recently, Anh [18] proposed a new way for determining the averaging values, instead of using the conventional averaging process, the author introduced weighted coefficient functions, and by this manner the averaging value is so-called the weighted averaging value. Anh's proposed method has been effectively applied to analyse some strongly nonlinear oscillations [19].

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In this paper, the equivalent linearization method with a weighted averaging value is used to achieve the periodic solutions for the motions of two-mass systems with linear and nonlinear springs. The solutions are compared with the ones given by Bayat and Pakar using the Hamiltionian Approach [1]. The results show accuracy of the present method for the considered nonlinear oscillating systems.

2. Review of the equivalent linearization method

To introduce an overview of the Equivalent Linearization method, one considers the oscillation described by following nonlinear differential equation:

$$\ddot{X} + g(X) = 0, \quad X(0) = A, \;\; \dot{X}(0) = 0 \qquad (1)$$

where *g(X)* is a nonlinear function of *X* and *A* is the initial amplitude.

The idea of the Equivalent Linearization method is to replace the nonlinear term *g(X)* in Eq. (1) by the linear term as follows:

$$g(X) \to \alpha X \qquad (2)$$

By this manner, the linearized equation of Eq. (1) is given by:

$$\ddot{X} + \alpha X = 0 \qquad (3)$$

where the cofficient *α* of the linear term is determined by using the mean-square criterion:

$$e(X) = g(X) - \alpha X \to \underset{\alpha}{Min} \qquad (4)$$

Thus, from:

$$\frac{\partial \left\langle e^2(X) \right\rangle}{\partial \alpha} = 0$$

yields:

$$\alpha = \frac{\left\langle g(X)X \right\rangle}{\left\langle X^2 \right\rangle} \qquad (5)$$

In Eq. (5), the symbol $\langle \cdot \rangle$ donotes the time-averaging operator in classical meaning [14]:

$$.\left\langle f(t) \right\rangle = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(t)dt \qquad (6)$$

For a *ω*-frequency function *f(ωt)*, the averaging process is taken during one period *T*:

$$\left\langle f(\omega t) \right\rangle = \frac{1}{T} \int_0^T f(\omega t)dt = \frac{1}{2\pi} \int_0^{2\pi} f(\tau)d\tau, \quad \tau = \omega t \qquad (7)$$

The everaging values in Eqs. (6) and (7) are called the classical or conventional averaging values. They often give incorrect results, especially for some periodic functions such as *sine* or *cosine* ones.

In this paper, the weighted averaging value proposed by Anh [18] is used to calculate averaging values in Eq. (5) instead of the conventional averaging values in Eq. (6) or Eq. (7). The idea of the proposed method as follows [18]: replacing the constant coefficient *1/T* in Eq. (6) by a weighted coefficient function *h(t)*. Thus, the averaging value is called a weighted averaging value:

$$\langle x(t)\rangle = \int_0^T h(t)x(t)dt \quad \text{.....................(8)}$$

where *h(t)* satisfies the condition:

$$\int_0^T h(t)dt = 1 \quad \text{.................. (9)}$$

In the Ref. [18], Anh has proposed a weighted function as follows:

$$h(t) = s^2\omega^2 t e^{-s\omega t}, s > 0 \quad \text{.................. (10)}$$

where *s* is constant.

It is seen that the weighted coefficient (10), obtained as a product of the optimistic weighted coefficient *t* and the pessimistic weighted coefficient $e^{-s\omega t}$, has one maximal value at $t_{\max} = 1/(\omega s)$, and then decreases to zero as $t \to \infty$. If one requires that the time t_{max} is equal to $T/n=2\pi/(n\omega)$ where *n* is a natural number or zero, we get $s=n/(2\pi)$. So the meaning of *s* can be specified as follows: for $n = 1$, $s=1/(2\pi)$ the weighted coefficient (10) has maximal value after one period, and for $n=4$, $s=4/(2\pi)$ the weighted coefficient (10) has maximal value after quarter period, and for $n=0$, $s=0$ the weighted coefficient (10) has maximal value at infinity, this case corresponds to the conventional averaging value. The detailed properties of the weighted function *h(t)* in Eq. (10) can be viewed in Refs. [18, 19].

With the priodic solution of linearized equation (3), the averaging values in Eq. (5) can be calculated by using Eq. (8).

3. Nonlinear two-degree-of-freedom oscillator system cases

3.1. Case 1

Figure 1 represents a two mass system connected by linear and nonlinear springs. The governing equation of motion is given as [1, 2]:

$$\begin{aligned} m\ddot{x} + k_1(x-y) + k_2(x-y)^3 &= 0 \\ m\ddot{y} + k_1(y-x) + k_2(y-x)^3 &= 0 \end{aligned}, \quad \text{.................. (11)}$$

with initial conditions

$$\begin{aligned} x(0) &= x_0, \quad \dot{x}(0) = 0 \\ y(0) &= y_0, \quad \dot{y}(0) = 0 \end{aligned}, \quad \text{..............(12)}$$

where k_1 and k_2 are the stiffnesses of the linear and nonlinear springs, respectively.

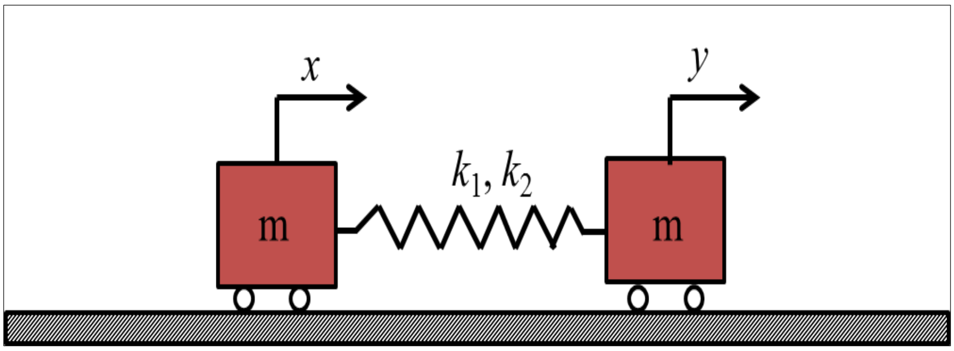

Figure 1 Two mass system connected by linear and nonlinear springs

Here, the new variables are introducted as follows:

$$X = x - y, \quad ....(13)$$

$$Y = x + y. \quad ....(14)$$

Transforming Eqs. (11) by using the new variables in Eqs. (13) and (14) yields:

$$\ddot{X} + 2\alpha X + 2\beta X^3 = 0, ...(15)$$

$$\ddot{Y} = 0, \quad ....(16)$$

with intial conditions

$$X(0) = x(0) - y(0) = x_0 - y_0 = A, \quad \dot{X}(0) = \dot{x}(0) - \dot{y}(0) = 0, \quad ....(17)$$

$$Y(0) = x(0) + y(0) = x_0 + y_0 = B, \quad \dot{Y}(0) = \dot{x}(0) + \dot{y}(0) = 0, \quad ....(18)$$

and parameters

$$\alpha = k_1 / m, \ \beta = k_2 / m.$$

It can be observed that Eq. (15) is the cubic Duffing equation, Eq. (16) can be solved independently. Now, one will apply the method introduced in the section 2 to solve Eq. (15), then we will find the solution of Eq. (16).

Firstly, the linearized equation of Eq. (15) is introduced as follows:

$$\ddot{X} + \omega^2 X = 0 ....(19)$$

The equation error between the two oscillators given in Eq. (15) and Eq. (19) is:

$$e(X) = 2\alpha X + 2\beta X^3 - \omega^2 X$$

where ω^2 is determined by using the mean square error criterion.

Thus, from:

$$\frac{\partial}{\partial \omega^2} \left\langle e^2 \right\rangle = 0$$

yields:

$$\omega^2 = \frac{2\alpha\left\langle X^2\right\rangle + 2\beta\left\langle X^4\right\rangle}{\left\langle X^2\right\rangle} \qquad \text{..............(20)}$$

The periodic solution of the linearized equation (25) is:

$$X = A\cos(\omega t)\,. \qquad \text{..........(21)}$$

With the solution in Eq. (21) and the weighted coefficient in Eq. (10), we will calculate $\left\langle X^2\right\rangle$ and $\left\langle X^4\right\rangle$ by using Eq. (8):

$$\left\langle X^2\right\rangle = \left\langle A^2\cos^2(\omega t)\right\rangle = A^2\frac{s^4+2s^2+8}{(s^2+4)^2}\,, \qquad \text{..................(22)}$$

$$\left\langle X^4\right\rangle = \left\langle A^4\cos^4\omega t\right\rangle = A^4\frac{248s^4+416s^2+1536+28s^6+s^8}{(s^2+4)^2(s^2+16)^2}\,. \qquad \text{...............(23)}$$

Substituting Eqs. (22) and (23) into Eq. (20), we obtain the approximate frequency:

$$\omega = \sqrt{2\alpha + 2\beta A^2\frac{248s^4+416s^2+1536+28s^6+s^8}{(s^4+2s^2+8)(s^2+16)^2}}\,. \qquad \text{..................(24)}$$

With the parameter *s* is chosen equal to 2, the approximate frequency will be:

$$\omega = \sqrt{2\alpha + 2\beta A^2\frac{9216}{12800}} = \sqrt{2\alpha + 1.44\beta A^2}\,. \qquad \text{..........(25)}$$

Therefore, the approximate solution for *X(t)* can be found:

$$X = A\cos\left(\sqrt{2\alpha+1.44\beta A^2}\,t\right). \qquad \text{...............(26)}$$

Now, we will find the solution of Eq. (16). The solution of Eq. (16) has the form:

$$Y = C_1 t + C_2\,. \qquad \text{..................(27)}$$

With the intial conditions (18), we find that

$$C_1 = 0, \quad C_2 = B\,. \qquad \text{................(28)}$$

Thus, the solution of Eq. (16) is

$$Y = B\,. \qquad \text{.............(29)}$$

Substituting approximate solutions in Eqs. (26) and (28) into Eqs. (13) and (14), the approximate solutions *x(t)* and *y(t)* are:

$$X = \frac{A}{2}\cos\left(\sqrt{2\alpha + 1.44\beta A^2}\, t\right) + \frac{B}{2}, \quad \text{..................(30)}$$

$$Y = -\frac{A}{2}\cos\left(\sqrt{2\alpha + 1.44\beta A^2}\, t\right) + \frac{B}{2}. \quad \text{..................(31)}$$

3.2. Case 2

Figure 2 represents a two mass system with linear and nonlinear springs connections fixed to the body. The governing equation of motion is [1, 3]:

$$\begin{aligned} m\ddot{x} + k_1 x + k_2(x-y) + k_3(x-y)^3 = 0, \\ m\ddot{y} + k_1 y + k_2(y-x) + k_3(y-x)^3 = 0. \end{aligned} \quad \text{................(32)}$$

with initial conditions:

$$\begin{aligned} x(0) = x_0, \quad \dot{x}(0) = 0 \\ y(0) = y_0, \quad \dot{y}(0) = 0 \end{aligned} \quad \text{..................(33)}$$

where k_1 and k_2 are the stiffnesses of the linear springs, and k_3 is the stiffness of nonlinear spring.

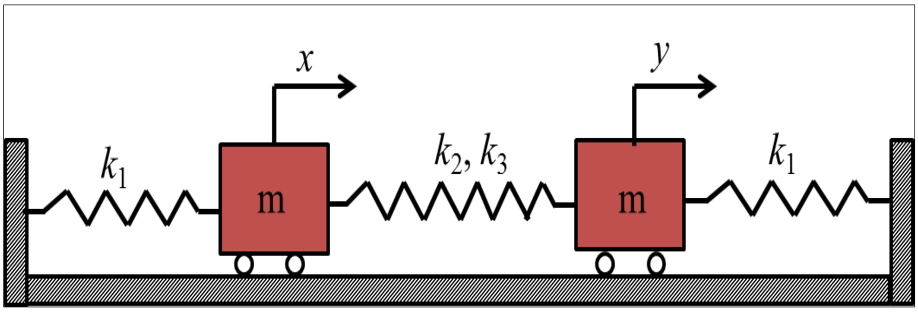

Figure 2 Two mass fixed-body-system with linear and nonlinear springs

As in Case 1, transforming Eqs. (32) by using the new variables in Eqs. (13) and (14) yields:

$$\ddot{X} + (\delta + 2\rho)X + 2\varepsilon X^3 = 0, \quad \text{....................(34)}$$

$$\ddot{Y} + \delta Y = 0. \quad \text{..................(35)}$$

with intial conditions:

$$X(0) = x(0) - y(0) = x_0 - y_0 = A, \quad \dot{X}(0) = \dot{x}(0) - \dot{y}(0) = 0, \text{ .... (36)}$$

$$Y(0) = x(0) + y(0) = x_0 + y_0 = B, \quad \dot{Y}(0) = \dot{x}(0) + \dot{y}(0) = 0. \text{ .... (37)}$$

and parameters:

$$\delta = k_1 / m, \ \rho = k_2 / m, \ \varepsilon = k_3 / m.$$

It is similar to Case 1, we will solve Eq. (34), and then find solution of Eq. (35).

The equivalent linear equation of Eq. (34) is given as follows:

$$\ddot{X} + \omega^2 X = 0. \qquad \text{....................(38)}$$

By using the mean square error criterion, we get:

$$\omega^2 = \frac{(\delta + 2\rho)\left\langle X^2 \right\rangle + 2\varepsilon \left\langle X^4 \right\rangle}{\left\langle X^2 \right\rangle}. \qquad \text{.......................(39)}$$

The periodic solution of the equivalent linear equation (38) is:

$$X = A\cos(\omega t). \qquad \text{..................(40)}$$

With the solution in Eq. (40) and the weighted coefficient in Eq. (10), we can calcualate $\left\langle X^2 \right\rangle$ and $\left\langle X^4 \right\rangle$ by using Eq. (8). The approximate frequency can be obtained:

$$\omega = \sqrt{(\delta + 2\rho) + 2\varepsilon A^2 \frac{248s^4 + 416s^2 + 1536 + 28s^6 + s^8}{(s^4 + 2s^2 + 8)(s^2 + 16)^2}}. \qquad \text{..................(41)}$$

With the parameter *s* is chosen equal to 2, the approximate frequency will be:

$$\omega = \sqrt{(\delta + 2\rho) + 2\varepsilon A^2 \frac{9216}{12800}} = \sqrt{(\delta + 2\rho) + 1.44\varepsilon\beta A^2}. \qquad \text{.....................(42)}$$

Therefore, the approximate solution for *X(t)* is:

$$X = A\cos\left(\sqrt{(\delta + 2\rho) + 1.44\varepsilon A^2}\, t\right). \qquad \text{...........................(43)}$$

And now, we will find the solution of Eq. (35). The solution of Eq. (35) has the form:

$$Y = C_1 \cos\left(\sqrt{\delta}\, t\right) + C_2 \sin\left(\sqrt{\delta}\, t\right). \qquad \text{...........(44)}$$

With the intial conditions (37), we find that:

$$C_1 = B, \quad C_2 = 0. \qquad \text{....................} \quad (45)$$

Thus, the approximate solution for *Y(t)* is:

$$Y = B\cos\left(\sqrt{\delta}\, t\right), \qquad \text{................(46)}$$

Finally, the approximate solutions *x(t)* and *y(t)* can be obtained as:

$$\begin{aligned} x(t) &= \frac{A}{2}\cos\left(\sqrt{(\delta + 2\rho) + 1.44\varepsilon A^2}\, t\right) + \frac{B}{2}\cos(\sqrt{\delta}\, t), \\ y(t) &= -\frac{A}{2}\cos\left(\sqrt{(\delta + 2\rho) + 1.44\varepsilon A^2}\, t\right) + \frac{B}{2}\cos(\sqrt{\delta}\, t). \end{aligned} \qquad \text{.................(47)}$$

4. Discusion cases

To illustrate and verify accuracy of the proposed method used in this work, the obtained solutions are compared with the published data and exact solutions. The exact frequency ω_e for a dynamical system governed by equation

$$\ddot{X} + \alpha X + \beta X^3 = 0.$$

can be derived as shown in Eq. (48) as follows [6]:

$$\omega_e(X_0) = \frac{\pi\sqrt{\alpha + \beta X_0^2}}{2}\left(\int_0^{\pi/2} \frac{dt}{1 - m\sin^2 t}\right)^{-1} \quad \text{..................(48)}$$

where:

$$m = \frac{\beta X_0^2}{2(\alpha + \beta X_0^2)}. \quad \text{...............(49)}$$

The approximate frequency ω_{HA} using the Hamiltionian Approach (HA) [1] is given as follows:

$$\omega_{HA} = \frac{1}{2}\sqrt{8\alpha + 6\beta A^2}, \quad \text{.................(50)}$$

for Case 1, and

$$\omega_{HA} = \frac{1}{2}\sqrt{4\delta + 8\rho + 6\varepsilon A^2}. \quad \text{.....................(51)}$$

for Case 2.

Table 1 and Table 2 provide comparisons of the present results and the HA results [1] with the exact ones for different values of m, k_1, k_2, k_3 and initial conditions x_0, y_0. The maximum relative error between the present results and the exact results is 0.4782%; while with the HA, the maximum relative error is 2.2124%. It is showed that the accuracy of the solutions obtained by the present method compared with the HA.

Table 1 Comparison of the frequencies of the system (*Case 1*)

Constant parameters					**Exact solution**	**HA solution**		**Present solution**	
m	k_1	k_2	x_0	y_0	ω_{ex}	ω_{HA}	**Error (%)**	$\omega_{present}$	**Error (%)**
1	2	3	1	2	2.8983	2.9155	0.5941	2.8844	0.4782
1	4	5	1	3	6.0823	6.1644	1.3501	6.0663	0.2631
2	5	3	-1	4	7.6838	7.8262	1.8534	7.6811	0.0345
2	8	6	-4	4	16.8517	17.2047	2.0946	16.8665	0.0880
5	5	5	5	15	12.0683	12.3288	2.1587	12.0830	0.1222
5	10	15	-5	10	31.1957	31.8826	2.2019	31.2410	0.1452
10	15	20	10	30	33.9348	34.6843	2.2087	33.9853	0.1488
20	40	50	15	40	47.4048	48.4536	2.2124	47.4763	0.1508
50	100	50	-20	10	36.0023	36.7967	2.2065	36.0555	0.1478

Table 2 Comparison of the frequencies of the system (*Case 2*)

Constant parameters						Exact solution	HA solution		Present solution	
m	k_1	k_2	k_3	x_0	y_0	ω_{ex}	ω_{HA}	Error (%)	$\omega_{present}$	Error (%)
1	0.5	0.5	0.5	1	2	1.4965	1.5000	0.2331	1.4900	0.4366
1	1	1	2	-1	1	3.8200	3.8730	1.3879	3.8105	0.2484
2	2	1	4	5	1	6.9300	7.0711	2.0357	6.9340	0.0573
2	4	3	5	-3	2	9.7436	9.9373	1.9882	9.7468	0.0328
5	10	5	10	2	10	13.7087	14.0000	2.1250	13.7230	0.1041
5	5	20	10	-15	-5	17.2194	17.5784	2.0850	17.2337	0.0830
10	10	20	5	5	20	12.9120	13.1814	2.0866	12.9228	0.0840
10	20	30	10	-10	15	30.0928	30.7490	2.1806	30.1330	0.1337
20	40	50	20	-20	10	36.0459	36.8375	2.1961	36.0971	0.1420

To further illustrate and verify accuracy of this new approximate analytical approach, comparisons of the time history of oscillatory displacement responses for the two masses with exact solutions are presented in Figures 3 and 4 (for Case 1) and Figures 5 and 6 (for Case 2). These figures show the time history of displacement responses of the systems with exact results for different parameters of the systems. The exact solution using Jacobian elliptic function developed by Cveticanin [3]. Apparently, it is confirmed that the present analytical approximations show excellent agreement with the exact solution using Jacobi elliptic function.

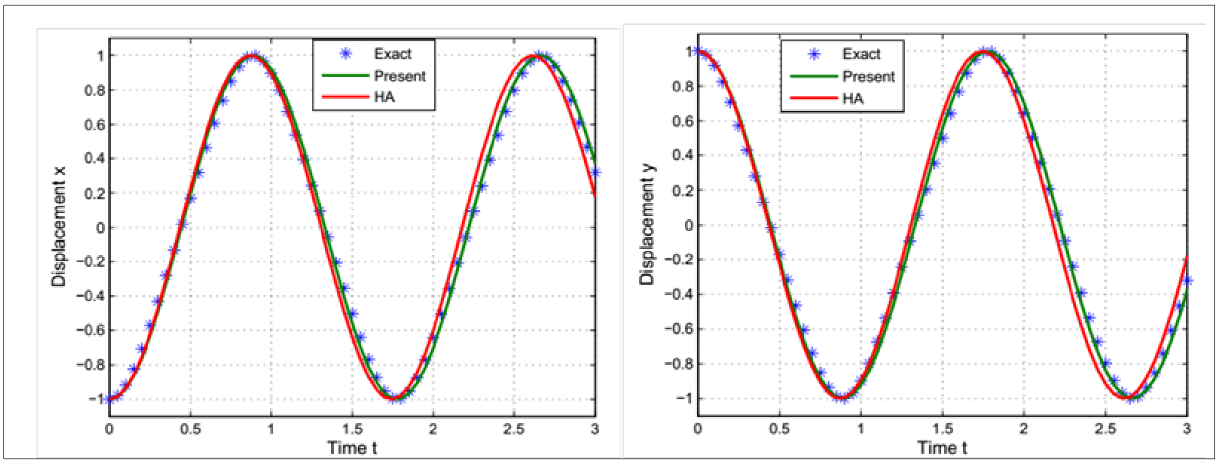

Figure 3 Comparison of the analytical approximate solutions with the exact solution for *m=1*, *k*$_1$*=1*, *k*$_2$*=2*, *x*$_0$*=-1*, *y*$_0$*=1*, (Case 1)

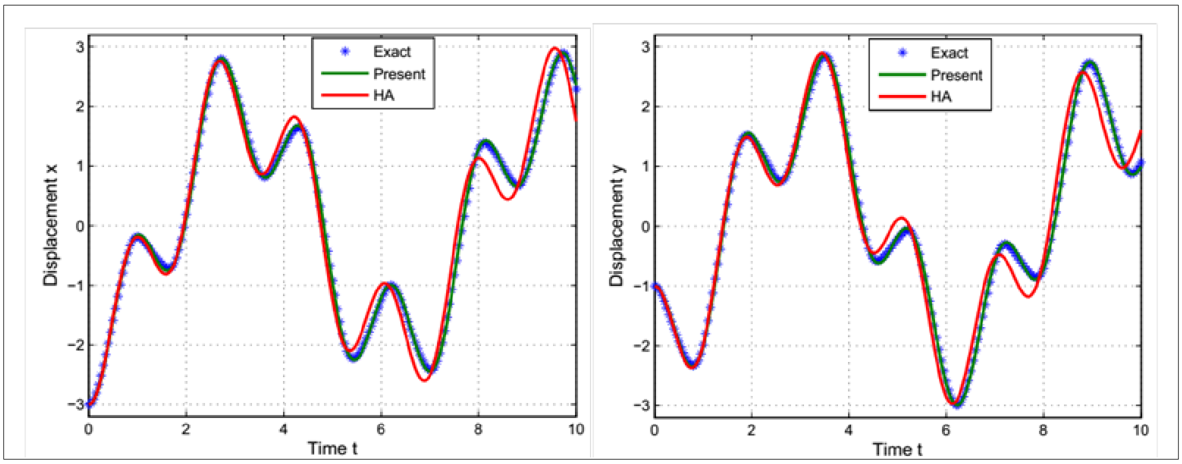

Figure 4 Comparison of the analytical approximate solutions with the exact solution for $m=1$, $k_1=1$, $k_2=2$, $x_0=-3$, $y_0=-1$, (Case 1)

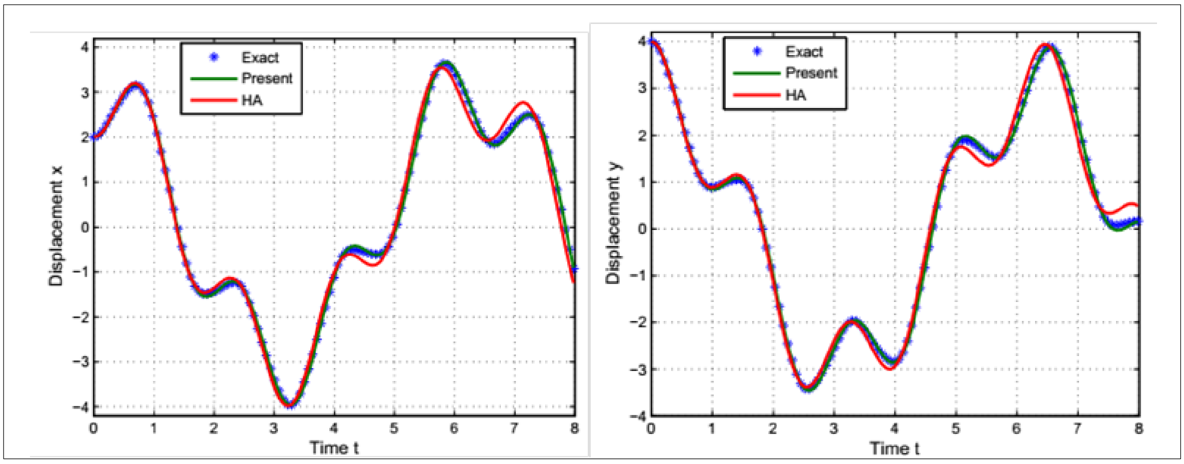

Figure 5 Comparison of the analytical approximate solutions with the exact solution for $m=1$, $k_1=1$, $k_2=1$, $k_3=2$, $x_0=2$, $y_0=4$, (Case 2)

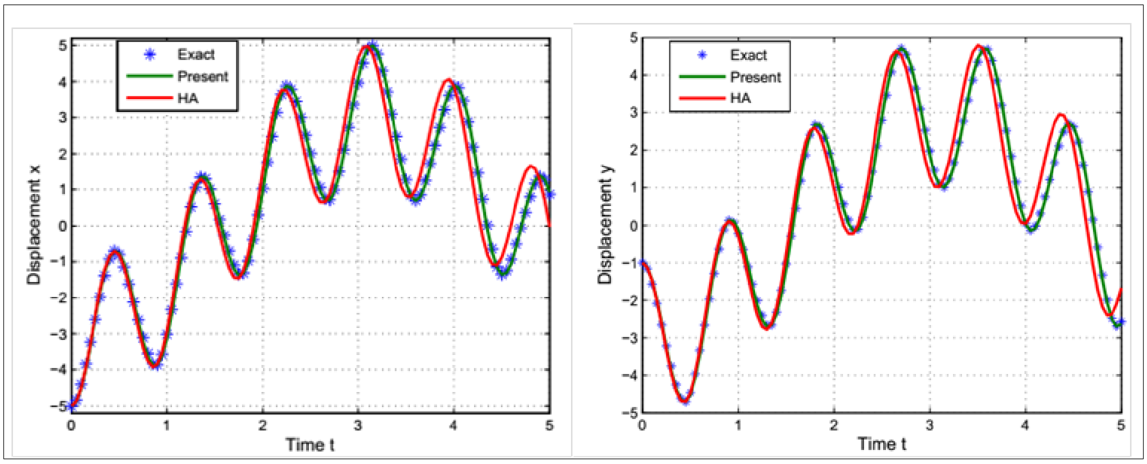

Figure 6 Comparison of the analytical approximate solutions with the exact solution for $m=1$, $k_1=1$, $k_2=1$, $k_3=2$, $x_0=-5$, $y_0=-1$, (Case 2)

5. Conclusion

In this paper, a new analytical method is developed for analysing the motions of the nonlinear two-degree-of-freedom oscillation systems having linear and nonlinear stiffnesses. Two practical examples of two-mass systems with free and fixed ends and with linear and nonlinear stiffensses are presented and discussed. The present solutions are compared with the ones achieved by using the HA. The maximum relative error of the HA results is 2.2124%, while the maximum relative error of the present method is only 0.4782%. The effects of different parameters on the response of the systems are shown graphically and numerically discussions. The proposed technique is considered as the combination of the classical equivalent linearization method with the weighted averaging concept. The results show the reliability of this method. Futhermore, this method can be applied for investigating other kinds of engineering oscillation in future researches.

Compliance with ethical standards

Acknowledgments

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