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Explicit identities of matrix powers of matrix

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Abstract

In this paper, we present some new explicit identities of matrix powers of matrix and their proofs. For instance, $A^{C+D} = A^C + A^D$ and $(A^B)^T = A^{B^T}$ for some matrices A, B and C.

Keywords: Principal matrix power; Matrix exponential; Matrix logarithm; Matrix power of matrix

1. Introduction

The exponential of matrix A is defined via its Taylor series, $e^A = I + \sum_{n\geq 1}\frac{1}{n!}A^n$ [1], and the matrix logarithmic is defined by $\log(A) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}(A-I)^n$, where $\|A - I\| < 1$ [2]. The principal matrix power A^α for a matrix $A \in \mathbb{C}^{n\times n}$ and a real number $\alpha \in \mathbb{R}$ is defined by $A^\alpha = \exp(\alpha\, log(A))$ [3]. The matrix power of A to B is defined by $A^B = \exp(B\log(A))$ [4]. In this paper, first present the properties of matrix exponential, and matrix logarithm. Finally, prove the explicit identities of matrix powers of matrix.

1.1. Properties of matrix exponential for more detail in [5]

Let $A, B \in M_n(\Bbbk)$.

(i) $\exp(0_{n\times n}) = I_{n\times n}$.
(ii) $B\exp(A) = \exp(A)B$, if $AB = BA$.
(iii) $\exp(A+B) = \exp(A) + \exp(B)$, if $AB = BA$.
(iv) $\exp(A^{-1}) = (\exp(A))^{-1}$
(v) $\exp(A^T) = (\exp(A))^T$
(vi) $\exp(A) = U\exp(D)U^{-1}$, where $A = UDU^{-1}$.

1.2. Properties of matrix logarithm [6]

Let $A, B \in N_{M_n(\Bbbk)}(I, 1)$, where $N_{M_n(\Bbbk)}(I,1) = \{A \in M_n(\Bbbk) | \|A - I\| < 1\}$.

(i) $\log(AB) = \log(A) + \log(B)$, if $AB = BA$.
(ii) $B\log(A) = \log(A)\mathrm{B}$, if $AB = BA$ and $\|B\| < 1$.
(iii) $\log(A^{-1}) = -\log(A)$.
(iv) $\log(A) = U\log(D)U^{-1}$, where $A = UDU^{-1}$.
(v) $\log(A^T) = (\log(A))^T$.

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Results (i), (iii), (iv) and (v) are well known [6]. Proof of the result (ii) is given below.

Proof of (ii): Suppose $BC = CB$ and $\|B\| < 1$. Since $\|B\| < 1$, $B\log(A) = \sum_{n\geq 1} B \frac{(-1)^{n-1}}{n} (A - I)^n$, and by mathematical induction $B(A - I)^n = (A - I)^n B$. Hence, $\left(\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} (A - I)^n\right) B = \log(A)\, B$. ■

1.3. The functions exp and log satisfy the following two properties [2]

- (i) If $\|A - I\| < 1$, then $\exp(\log(A)) = A$ and;
- (ii) If $\|\exp B - I\| < 1$, then $\log(\exp(B)) = B$.

1.4. Explicit identites of matrix power matrix

Let $A \in N_{M_n(\mathbb{k})}(I, 1)$ and $B, C \in GL_n(\mathbb{k})$, where $\mathbb{k} = \mathbb{R}$, the real numbers, or $\mathbb{k} = \mathbb{C}$, the

complex numbers.

- (i) $A^{0_{n\times n}} = I_{n\times n}$.
- (ii) $A^{B+C} = A^B A^C$, if $BC = CB, BA = AB$ and $\|B\| < 1$.
- (iii) $A^{-I} = (A^{-1})^I$.
- (iv) $A^{B^T} = (A^B)^T$, if A is symmetric, $BC = CB$ and $\|B\| < 1$. .
- (v) $A^B = UD^B U^{-1}$, where $D = UAU^{-1}$.

Proof of (i):

$$A^{0_{n\times n}} = \exp(0_{n\times n} \log(A))$$

$$= \exp(0_{n\times n})$$

$$= I_{n\times n}. ■$$

Proof of (ii):

Suppose $BC = CB$ and $\|B\| < 1$.

$$A^{B+C} = \exp\big((B + C) \log(A)\big).$$

$$= \exp(B \log(A) + C \log(A)).$$

$$= \exp(B \log(A)) . \exp(C \log(A)). \text{ (by 1. (ii) and 2. (ii))}$$

$$= A^B A^C. ■$$

Proof of (iii):

$$A^{-I} = \exp((-I) \log(A)).$$

$$= \exp(I(-\log(A))).$$

$$= \exp(I \log(A^{-1})). \text{ (by 2. (iii))}$$

$$= (A^{-1})^I. ■$$

Proof of (iv):

Suppose A is symmetric, $BC = CB$ and $\|B\| < 1$.

$$A^{B^T} = \exp(B^T \log(A)).$$

$= \exp(B^T \log(A^T))$.

$= \exp(B^T (\log(A))^T)$. (by 2. (v))

$= \exp(((\log(A))B)^T)$.

$= \exp((B(\log(A)))^T)$. (by 2. (ii))

$= (\exp(B \log(A)))^T$. (by 1. (v))

$= (A^B)^T$. ■

Proof of (v):

Suppose $BU = UB$.

$A^B = exp\big(Blog(A)\big)$.

$= exp\big(Blog(UDU^{-1})\big)$.

$= exp(B(Ulog(D)U^{-1}))$.

$= exp(U(Blog(D))U^{-1})$.

$= exp(U(\mathcal{D})U^{-1})$, where $\mathcal{D} = Blog(D)$.

$= U(exp(\mathcal{D}))U^{-1}$ where $\mathcal{D} = Blog(D)$.

$= U(exp\big(Blog(D)\big)U^{-1}$.

$= UD^BU^{-1}$. ■

2. Conclusion

In this work, we show that some matices rules and indeces rules are hold for the matrix powers of matrix.

Compliance with ethical standards

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