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Algebraic Structure of Matrix Powers of Matrix

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Abstract

This paper investigates the exponential of a matrix and its inverse operation, the matrix logarithm. The matrix exponential plays a fundamental role in connecting Lie algebras with matrix Lie groups, while the logarithm provides a formal inverse in a suitable neighborhood of the identity matrix. We establish fundamental properties of these functions, construct a group structure based on generalized matrix powers, and demonstrate its isomorphism with the exponential matrix group. This research highlights the structural and algebraic significance of the exponential and logarithmic functions in the context of matrix theory.

Keywords: Matrix exponential; Matrix logarithm; Exponential group; Matrix powers; Matrix groups.

1. Introduction

The main aim of this endeavor is to identify the algebraic structure of the matrix powers of the matrix. The exponential of a matrix is central to the theory of Lie groups, acting as the bridge between a Lie algebra and its associated matrix Lie group. It encodes structural information from the algebra into the group. Conversely, the logarithm of a matrix provides, within certain constraints, an inverse to the exponential function. Together, these two operations establish algebraically sophisticated framework for studying linear transformations, operator theory, and group structures.

In this paper, we define the matrix exponential and logarithm, prove their basic properties, and construct an abelian group structure from generalized matrix powers. Finally, we establish an isomorphism between this group and the matrix exponential group.

2. The Exponential of a Matrix

Let X be an $n \times n$ matrix, i.e., $X \in M_n(K)$, where $K = \mathbb{R}$ or $K = \mathbb{C}$. We define the exponential of X, denoted e^X or $exp(X)$ by the usual power series:

$$e^X = \sum_{m=1}^{\infty} \frac{X^m}{m!},$$

where $X^0 = I$, and I is the identity matrix of order n.

For any $X \in M_n(K)$, we define the norm:

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$$\| X \| = \sqrt{\left(\sum_{j,k}^{n} |X_{jk}|^2\right)}.$$

Proposition 1 ([1], Proposition 2.2)

Let , $Y \in M_n(\mathbb{K})$. Then:

(i). $e^0 = I$.

(ii). $(e^X)^* = e^{X^*}$

(iii). If X invertible, then e^X invertible and $(e^X)^{-1} = e^{-X}$

(iv). $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$ for all $\alpha, \beta \in \mathbb{C}$.

(v). If $XY = YX$, then $e^{X+Y} = e^X e^Y = e^Y e^X$.

(vi). If C invertible, then $e^{CXC^{-1}} = Ce^X C^{-1}$.

3. The Matrix Logarithm

The logarithm of a matrix is defined as the inverse of the matrix exponential, as far as possible. Let:

$A \in N_{M_n(\mathbb{R})}(I, 1) = \{A \in M_n(\mathbb{R}) |\ \|A - I\| < 1\}$.

Then, for $A \in N_{M_n(\mathbb{R})}(I, 1)$ the logarithm is given by the power series,

$$\log (A) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (A - I)^n.$$

Proposition 2

Let $A, B \in N_{M_n(\mathbb{R})}(I, 1)$. If $AB = BA$ and $\|B\| < 1$, then $\mathrm{B} \log(A) = \log(A)\, B$.

Proof.

Since $\|B\| < 1$,

$$\mathrm{B} \log(A) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} B(A - I)^n ,$$

By mathematical induction $B(A - I)^n = (A - I)^n B$. Hence,

$$\left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (A - I)^n\right) B = \log(A)\, B. \text{Thus}, B \log(A) = B \log(A) \blacksquare$$

4. Matrix Powers

Definition 1 ([2])

Let $A \in N_{M_n(\mathbb{R})}(I, 1)$ and $B \in M_n(\mathbb{R})$. The matrix power of A to B is defined as:

$$A^B = e^{B \log(A)}.$$

Proposition 3

Let $A \in N_{M_n(\mathbb{R})}(I,1)$ and $B, C \in M_n(\mathbb{R}), \|B\| < 1$ and $\|C\| < 1$.

(i). $A^{0_{n\times n}} = I_{n\times n}$.

(ii). If $BC = CB$, then $A^{B+C} = A^B A^C$.

Proof:

(i). $A^{0_{n\times n}} = \exp(0_{n\times n}\log(A))$

$= \exp(0_{n\times n})$

$= I_{n\times n}$. ■

(ii). By definition

$$A^{B+C} = e^{(B+C)\log(A)} = e^{B\log(A)+C\log(A).}$$

By proposition 2,

$$B\log(A)\,C\log(A) = \log(A)\,BC\log(A) = \log(A)\,CB\log(A) = C\log(A)\,B\log(A)$$

By proposition 1(v),

$e^{B\log(A)+C\log(A)} = e^{B\log(A)}e^{C\log(A)} = A^B A^C$. Thus, $A^{B+C} = A^B A^C$. ■

5. Group Structures

Definition 2

Definition 2. A group is a set G together with a map $*\colon\ G \times G \rightarrow G$

$[*\,(g_1, g_2) \longmapsto g_1 * g_2]$ satisfying the following four conditions.

- $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ for all $g_1,\ g_2,\ g_3 \in G$;
- There exists $e \in G$ such that $e * g = g * e = g$ for each $g \in G$;
- For each $g \in G$ there exists $h \in G$ such that $h * g = g * h = e$.
- G is abelian if $h * g = g * h$ for all $h, g \in G$.

Theorem 1

The set $ExpM_n(\mathbb{R}) = \{e^X | X \in M_n(\mathbb{R}), SX = XS \text{ for all } S \in M_n(\mathbb{R})\}$ is a group under the operation $e^X e^Y = e^{X+Y}$ for all $e^X,\ e^Y \in ExpM_n(\mathbb{R})$.

Proof:

Clearly, $I \in ExpM_n(\mathbb{R})$. If $e^X, e^Y \in ExpM_n(\mathbb{R})$, then $SX = XS$ for all $S \in M_n(\mathbb{R})$. Therefore, $YX = XY$, and hence, $e^X e^Y = e^{X+Y}$ for all $e^X, e^Y \in ExpM_n(\mathbb{R})$ (by Proposition1(v)).

Since $S(X+Y) = SX + SY = XS + YS = (X+Y)S$ for all $S \in M_n(\mathbb{R})$, $e^{X+Y} \in ExpM_n(\mathbb{R})$. Matrix addition is associative gives the associative property hold in $ExpM_n(\mathbb{R})$. By Proposition1 (iii), every element in $ExpM_n(\mathbb{R})$ is invertible. Proposition 1(v), $ExpM_n(\mathbb{R})$ is abelian. Hence $ExpM_n(\mathbb{R})$ is an abelian group. ■

In this study, we define $N_{M_n(\mathbb{R})}(I,1)$ as the set $\{A \in M_n(\mathbb{R}))| \|A - I\| < 1\}$. Henceforth, we will use $N_{M_n(\mathbb{R})}(0,1)$ to denote the set $\{X \in M_n(\mathbb{R}))| \|X\| < 1\}$. That is, $N_{M_n(\mathbb{R}))}(0,1) = \{X \in M_n(\mathbb{R}))| \|X\| < 1\}$.

Theorem 2

Let $A \in N_{M_n(\mathbb{R})}(I,1)$. The set

$$G = \left\{A^X \mid X \in M_n(\mathbb{R}), \exists k \in \mathbb{N} \text{ such that } X = \sum_{i=1}^{k} X_i \text{, where } \forall\, i, X_i \in N_{M_n(\mathbb{R}))}(0,1), \text{and } \forall\, T \in N_{M_n(\mathbb{R}))}(0,1), X_iT = TX_i\right\}$$

is a group under the binary operation $A^XA^Y = A^{X+Y}$ for all $A^X, A^Y \in G$.

Proof:

Let $A \in N_{M_n(\mathbb{R})}(I,1)$. $A^0 = I \in G$ because, $\| 0_{n\times n} \| = 0 < 1$ and $XT = 0_{n\times n}T = T0_{n\times n} = TX$ for all $T \in N_{M_n(\mathbb{R}))}(0,1)$, hence G is non-empty.

Let $A^X, A^Y \in G$. Then $X, Y \in M_n(\mathbb{R})$ and $\exists k, l \in \mathbb{N}$ such that

$$X = \sum_{i=1}^{k} X_i, Y = \sum_{i=1}^{l} Y_i, \text{where } X_i, Y_i \in N_{M_n(\mathbb{R}))}(0,1) \text{ for all } i \text{ and } X_iT = TX_i \text{ and } Y_iT = TY_i \text{ for all } T \in N_{M_n(\mathbb{R}))}(0,1).$$

Therefore, $X_iY_i = Y_iX_i$ for all $i = 1,2,\dots,t$, where $t = Max\{k,l\}$. Hence, $XY = YX$.

By proposition 2,

$$Xlog(A) = \sum_{i=1}^{k} X_i \log(A) = \log(A)\sum_{i=1}^{k} X_i = \log(A)X,$$

and similarly, $Ylog(A) = \log(A)Y$. Therefore, $Xlog(A)Ylog(A) = Ylog(A)Xlog(A)$, and by proposition 3 (ii) we have,

$$A^XA^Y = A^{X+Y} \text{and } X + Y = \sum_{i=1}^{k+l} Z_i,$$

where $Z_i = X_i$ for all $i = 1,2,\dots,k$ and $Z_i = Y_i$ for all $i = k+1, k+2, \dots, k+l$, and $(X+Y)T = XT + YT = TX + TY = Y(X+Y)$. Therefore, $A^{X+Y} \in G$. Thus, for all $A^X, A^Y \in G$, $A^XA^Y \in G$. Matrix addition is associative gives the associative property hold in G. Identity of G is $I = A^0$, and $A^{-X} \in G$ is the inverse of any $A^X \in G$. Since $X + Y = Y + X$, G is an abelian group. ∎

Theorem 3

Group G is isomorphic to $ExpM_n(\mathbb{R})$.

Proof:

Define $\phi: G \longrightarrow ExpM_n(\mathbb{R})$ by $\phi(A^X) = e^X$ for all $e^X \in G$. Then ϕ is a well-defined onto homomorphism. $Ker\phi = I$, hence, $G \cong ExpM_n(\mathbb{R})$. ∎

6. Conclusion

In this paper, we studied the matrix exponential and logarithm, highlighting their roles in the algebraic structure of matrix groups. We introduced generalized matrix powers, proved their fundamental properties, and constructed a group (G) that is abelian and isomorphic to the exponential group of matrices. These results underscore the deep interplay between exponential and logarithmic functions in matrix theory, offering insight into applications across Lie theory, operator analysis, and linear algebra. Future work may extend these results to infinite-dimensional operator algebras and explore computational aspects in applied settings.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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