World Journal of Advanced Engineering Technology and Sciences
eISSN: 2582-8266
Cross Ref DOI: 10.30574/wjaets
Journal homepage: https://wjaets.com/

(Research Article)

Check for updates

Covering System with Restricted Moduli: Theory, Existence and Computational Structure

Olalekan Peter Ogunmefun *

Department of mathematics, university of South Carolina, Columbia, SC 29208, USA.

World Journal of Advanced Engineering Technology and Sciences, 2025, 17(03), 179–196

Publication history: Received 26 October 2025; revised on 08 December 2025; accepted on 10 December 2025

Article DOI: https://doi.org/10.30574/wjaets.2025.17.3.1549

Abstract

We study covering systems whose moduli are restricted to specific multiplicative families, including primes, prime powers, and smooth numbers. Combining analytic tools such as reciprocal-sum inequalities and sieve obstructions with computational methods using SAT solvers, ILP, and CRT-based search, we obtain explicit nonexistence results for prime-only covering systems, construct new examples for prime-power families, and provide extensive evidence for threshold behavior in smooth-moduli families. Our work unifies theoretical and algorithmic approaches, yielding new insights into covering systems under multiplicative constraints and suggesting several directions for future research.

Keywords: Covering System; Moduli; Prime Power; Smooth Number; Chinese Remainder Theorem

1. Introduction

A *covering system* of the integers is a finite set of congruences

$$\{a_i \quad (mod n_i)\}_{i=1}^{k},$$

such that every integer m satisfies at least one congruence $m \equiv a_i \pmod{n_i}$. If the moduli n_i are all distinct and greater than 1, the system is called a *distinct covering system*. Covering systems were introduced by Erdo˝s in the 1950s as part of his investigations into the additive and combinatorial structure of the integers [4].

Over the decades, numerous constructions and structural results for covering systems have appeared. Early work of Selfridge [12] and others constructed explicit covering systems and highlighted their combinatorial richness. A long-standing question posed by Erd˝os asked whether distinct covering systems could have arbitrarily large minimum modulus. This was finally resolved by Hough [9], who proved that the minimum modulus in any distinct covering system is bounded above by an absolute constant. Hough's result introduced powerful probabilistic and density-based techniques into the study of coverings.

In recent years, significant progress has been made on understanding the deeper structure of covering systems. The work of Filaseta, Ford, Konyagin, Pomerance, and Yu [7] developed sieve-theoretic and reciprocal-sum obstructions showing that many families of moduli are too sparse to support a distinct covering. Modern surveys and structural analyses by Balister, Bolloba´s, Morris, Sahasrabudhe, and Tiba [1, 2] further illustrate that covering systems form a surprisingly rigid combinatorial object and connect classical results with contemporary methods in probabilistic number theory. Sahasrabudhe [11] provides additional recent insights into the emerging landscape of covering system theory.

* Corresponding author: Olalekan ogunmefun; Email: olalekan@email.sc.edu

Most classical results on covering systems make no structural assumptions on the moduli n_i beyond distinctness. However, it is natural and increasingly important to explore the behavior of covering systems when the set of allowed moduli is restricted to a specific multiplicative family. Examples include; primes up to a parameter P, prime powers bounded by P and y-smooth integers (those with all prime factors at most y)

Studying restricted-moduli covering systems is motivated by several factors. First, such restrictions create a more structured setting in which analytic tools such as sieve methods and reciprocal-sum bounds can be deployed with greater precision. For instance, prime moduli enjoy well-understood distribution properties, enabling explicit bounds for sums such as $\sum_{p\leq P} 1/p$, which are relevant to covering feasibility.

Second, restricting the moduli significantly reduces the search space for algorithmic investigation. Computational methods based on the Chinese Remainder Theorem (CRT), backtracking with pruning, and SAT/ILP formulations—become more tractable in these settings. Finally, restricted families provide conceptual bridges to other areas of number theory, including multiplicative combinatorics and the theory of smooth numbers.

Despite the naturality of these questions, the restricted-family viewpoint remains relatively underdeveloped. While the classical literature offers powerful tools for the general setting [7, 9], few works systematically investigate coverings where the moduli are confined to primes, prime powers, or smooth sets. Our work addresses this gap.

In this paper, we undertake a systematic study of covering systems whose moduli belong to specific restricted multiplicative families. We derive new obstruction theorems for restricted families of moduli, especially the prime-only and prime-power cases. These results adapt reciprocal-sum and sieve-theoretic arguments of Filaseta et al. [7] to constrained settings, yielding explicit impossibility criteria. We develop an efficient computational methodology employing CRT-based residue decomposition, pruning-enhanced backtracking, and SAT/ILP encodings. This framework enables exhaustive exploration of restricted-moduli coverings for explicit ranges of parameters. By combining theoretical results with computational experiments, we obtain explicit threshold values beyond which coverings cannot exist for certain restricted families. When coverings do exist, we document structural patterns in their residue assignments. Drawing on both analytic and computational evidence, we formulate conjectures about the asymptotic behavior of restricted-family covering systems and discuss potential extensions to algebraic number fields and function fields.

This paper is organized as follows. In Section 2 we recall basic facts about covering system and Chinese remainder theorem (CRT). Section 3 develops the restricted family covering systems while Section 4 contains the theoretical obstructions. Section 5 gives the constructive methods and algorithmic framework and Section 6 illustrates the computational experiment. We conclude in Section 7 with the main theorem and conjectures for further research.

2. Preliminaries and Notation

This section establishes the terminology, notation, and mathematical tools used throughout the paper. We introduce restricted families of moduli, describe the Chinese Remainder Theorem (CRT) framework in which our constructions and computations are carried out, and briefly review the classical results that underpin the theoretical developments of later sections.

Given a collection of moduli $S = \{n_1, \dots, n_k\}$, we denote by

$$L = \mathrm{lcm}(n_1, \dots, n_k)$$

the least common multiple of the moduli. A covering system C covers Z if and only if it covers all residues modulo L.

Throughout this paper, the following restricted families of moduli will be used:

$\mathcal{P}(P) = \{p : p \text{ prime},\ p \leq P\}$, the set of primes up to P.

$\mathcal{P}^*(P) = \{p^e : p \leq P,\ p^e \leq P\}$, the set of prime powers bounded by P.

$\mathcal{S}(y) = \{n :$ all prime divisors of n are $\leq y\}$, the set of y-smooth integers.

When the allowed moduli come from one of these families, we refer to the system as a *restricted-family covering system*.

The Chinese Remainder Theorem plays a central role in both the theoretical and computational aspects of this work. For each modulus $n_i \in S$, the congruence a_i (mod n_i) corresponds to a subset of residues modulo L, namely the set

$$R(a_i, n_i) = \{r \in Z/LZ : r \equiv a_i (mod n_i)\}$$

Thus, a covering system is equivalent to finding a family of such subsets whose union is all of Z/LZ.

If the moduli factor as

$$n_i = \prod_{p \in \mathcal{P}(P)} p^{e_{i,p}}$$

then the CRT gives a canonical decomposition:

$$Z/LZ \cong \prod_{p \in P(P)} Z/p^{e_p Z}$$

where $L = \prod_{p \in \mathcal{P}(P)} p^{e_p}$ is the prime-power factorization of the least common multiple.

This decomposition is crucial for two reasons; it reduces coverage verification to a product structure, enabling combinatorial pruning and symmetry reductions and it allows efficient computation of coverage via SAT/ILP encodings, where residues are represented component- wise modulo each p^{e_p}.

Early constructions by Moser [10] and Selfridge [12] demonstrated the existence of covering systems with surprising and sometimes highly irregular structures. Erdo˝s posed several foundational conjectures, most notably that distinct covering systems might exist with arbitrarily large minimum modulus [5]. This conjecture was resolved negatively by Hough [9], who proved the existence of an absolute bound on the least modulus in a distinct covering system. Filaseta, Ford, Konyagin, Pomerance, and Yu [6] developed a powerful sieve-based framework for ruling out covering systems by partitioning moduli according to the size of their largest prime factors. Their techniques yield explicit lower bounds on the density of integers not covered by a proposed system, or equivalently, show that a collection of moduli cannot cover all of Z if the sum of reciprocals of moduli is too small. These results are central to our analysis, as restricted families such as $\mathcal{P}(P)$ and $\mathcal{S}(y)$ naturally produce small reciprocal sums, suggesting strong obstructions to coverage.

While coverage is a purely number-theoretic property, deciding whether a given set of moduli can form a covering is a finite combinatorial search problem modulo L. Recent work in computational number theory shows that SAT solvers, ILP tools, and symbolic CRT-based bitset manipulations can efficiently explore nontrivial search spaces, even for rapidly growing L. This computational perspective complements the analytic and sieve-theoretic techniques, and forms a key component of the present work.

The interplay of these classical theoretical results and modern computational approaches motivates a systematic study of covering systems with restricted moduli, which exhibit both interesting number-theoretic structure and algorithmic complexity.

3. Restricted-Family Covering Systems

In this section we introduce the families of moduli that will be the focus of the restricted- covering-system framework developed in this work. Restricting the moduli of a covering system to special multiplicative families produces a rich spectrum of phenomena that differ substantially from the classical unrestricted setting. These restrictions affect not only the combinatorial structure of residue classes but also the analytic quantities such as the sum of reciprocals of moduli that play a central role in sieve-theoretic obstruction results. We consider three principal families: primes, prime powers, and smooth numbers. Each family presents unique structural constraints and raises different existence and nonexistence questions for covering systems.

Prime-Only Moduli. For a real parameter $P \geq 2$, define the set of allowable moduli

$$\mathcal{P}(P) = \{p \in N : p \text{ prime}, \ p \leq P\}$$

A *prime-only covering system* is a covering system whose moduli all lie in P(P). Prime-only moduli are arguably the most restrictive natural family from the standpoint of multiplicative structure. Their relevance stems from several considerations;

The growth of the reciprocal sum

$$\sum_{p\leq P}\frac{1}{p} = \log\log P + B + o(1)$$

is extremely slow; consequently, sieve-theoretic methods such as those in [6] suggest that covering systems composed solely of primes may suffer from severe density limitations.

Unlike general moduli, primes allow exactly one residue class with each "period," reducing the combinatorial flexibility typically exploited in constructive CRT methods.

Prime moduli offer a clean, uniform structure that simplifies the theoretical analysis and facilitates computational exploration using SAT/ILP encodings.

The prime-only problem raises several natural questions:

- For which values of P does there exist a covering system with moduli in $\mathcal{P}(P)$?
- Is there a maximal value of P for which such a covering system exists?
- Can sieve-theoretic or reciprocal-sum methods rule out coverings for large P?
- How does the complexity of the search space grow with P? These questions form one of the central themes of the present work.

Prime-Power Moduli. The second family expands the allowable moduli to include prime powers. For a real parameter $P \geq 2$, define

$$\mathcal{P}^*(P) = \{p^e : p \text{ prime},\ p^e \leq P,\ e \geq 1\}$$

Prime-power covering systems have several features that distinguish them from the prime-only case:

Each modulus p^e admits p^e possible residue classes, offering greater combinatorial freedom for constructing coverings.

The CRT structure becomes richer, as higher powers of primes introduce deeper residue hierarchies.

The reciprocal sum $\sum_{p^e\leq P}\frac{1}{p^e}$ grows faster than potentially enabling fuller coverage.

Prime-power families are thus a promising candidate for constructing covering systems beyond the ranges where prime-only coverings fail. Nevertheless, they remain largely unexplored, and no systematic study of their covering behavior appears in the literature.

The key questions include the following;

- For which P do prime-power coverings exist?
- Does allowing prime powers extend the maximal threshold for possible coverings?
- Can theoretical obstructions similar to those of [6, 9] be adapted to this family?
- How do prime-power coverings compare computationally to prime-only coverings?

Smooth-Moduli Systems

The third family consists of y-smooth numbers:

$$\mathcal{S}(y) = \{n \in N : p \mid n \backslash implies p \leq y\}$$

Smooth numbers play an important role in analytic number theory, especially in the distribution of integers with restricted prime factors, friable numbers, and applications in sieve theory [13]. They interpolate naturally between prime-only and unrestricted moduli; for y = 2, the family consists of powers of 2, or moderate y, the family includes numbers with limited factorization complexity and as y grows, $\mathcal{S}(y)$ approaches the set of all positive integers. The main questions for smooth-moduli systems are:

- How does the value of y affect the feasibility of constructing coverings?
- Are there threshold phenomena as y varies?
- Can one characterize the minimal y for which covering becomes possible for arbitrarily large ranges?
- What computational strategies are effective for exploring these families?

These questions connect multiplicative structure, analytic density, and computational complexity in novel ways. Together, the three restricted families introduced in this section establish a framework for the theoretical and experimental investigations undertaken in the remainder of this paper.

4. Theoretical obstructions

In this section we develop analytical and combinatorial tools that yield explicit obstructions to the existence of restricted family covering systems. Our approach draws on three complementary methods:

- reciprocal-sum constraints, generalizing the framework of [6];
- sieve-theoretic lower bounds on the density of integers not covered by a proposed system [8];

counting and entropy arguments derived from the Chinese Remainder Theorem [3]. Together, these methods provide quantitative conditions under which a family of moduli cannot support a covering of Z, with particular emphasis on prime-only and prime-power families introduced in Section 3.

Reciprocal Sum Constraints. A classical heuristic suggests that a covering system with moduli $n_1, \ldots, n_k$ should require the sum of reciprocals $\sum_i 1/n_i$ to exceed 1, as each congruence a_i (mod n_i) covers exactly a $1/n_i$ proportion of the integers. However, the congruence classes overlap, and the reciprocal sum alone is neither necessary nor sufficient for a covering. Nevertheless, refined reciprocal-sum inequalities can be used to rule out coverings in restricted settings. The following lemma generalizes the obstruction framework of [6].

Lemma 4.1 (Reciprocal-Sum Obstruction)**.** *Let S be a set of distinct moduli and let*

$$R(S) = \sum_{n \in S} \frac{1}{n}$$

Suppose there exists $\delta > 0$ such that for every subset $T \subseteq S$ with least modulus ≥ M, the sum satisfies $R(T) < \delta$. Then no covering system can be formed using only moduli from S with all moduli ≥ M.

Proof. Let $L = \mathrm{lcm}(S)$, and identify integers with residues modulo L. Each residue class a (mod n) contributes exactly L/n residues modulo L. If C = {a_n (mod n) : $n \in S$} were a covering system, its union must cover all L residues.

By subadditivity of cardinalities,

$$L = \left| \bigcup_{n \in S} R(a_n, n) \right| \leq \sum_{n \in S} |R(a_n, n)| = LR(S)$$

Thus $R(S) \geq 1$ is necessary for converage. If all moduli are at least M and $R(S) < \delta < 1$, then the union covers at most δL < L residues, contradicting the assumption that C covers Z

In the restricted-family setting, this yields immediate consequences:

Corollary 4.2. *For prime-only moduli $\mathcal{P}(P)$, if*

$$\sum_{p \leq P} \frac{1}{p} < 1,$$

then no covering system exists using only primes ≤ P.

Since the sum of reciprocal primes is asymptotic to log log $P + B$, Corollary 4.2 does not directly forbid large coverings, but it provides necessary conditions that interact with deeper sieve constraints below.

Sieve-Theoretic Lower Bounds on Uncovered Density. Filaseta, Ford, Konyagin, Pomerance, and Yu [6] constructed a powerful "filtering" method based on grouping moduli according to the size of their largest prime factors. We adapt their perspective to restricted families. Let C be a candidate covering system with moduli S. Define the uncovered set

$$U = \{m \in Z/LZ : m \notin \bigcup_{n\in S} R(a_n, n)\}$$

The key idea is that residue classes modulo small primes dominate the coverage behavior. If many moduli share only small prime factors, their residue classes tend to overlap in structured patterns, leaving a positive density of integers uncovered.

The following theorem (a simplified restricted-family formulation) captures this.

Theorem 4.3 (Sieve Obstruction)**.** *Let S be a set of distinct moduli with* $P^+(n) \le y$ *for all*

$n \in S$*, where* $P^+(n)$ *denotes the largest prime factor of n. Let*

$$\Delta = \prod_{p\le y}\left(1-\frac{1}{p}\right)$$

If $\sum_{n\in S}\frac{1}{n} < \Delta^{-1}$ *then no covering system can be formed using the moduli in S.*

Proof. The filtering method of [6] groups moduli by their largest prime factors and applies a sieve to bound from below the density of residues not eliminated by any modulus. The product Δ is the natural sieve constant corresponding to integers free of prime factors ≤ y. If the total weighted coverage contribution $\sum 1/n$ is smaller than the sieve threshold Δ^{-1}, the uncovered density must remain positive, implying the union of residue classes cannot cover all of Z. Full technical details follow the method of [6]. ◻

For prime-only moduli with $S = \mathcal{P}(P)$, all moduli have largest prime factor P, so setting $y = P$ yields an explicit obstruction. For smooth-number systems $S \subseteq \mathcal{S}(y)$ the theorem gives stronger results due to the small size of Δ [11].

CRT-Based Counting and Entropy Arguments. The CRT decomposition of Section 2 allows us to treat coverage modulo L = lcm(S) as a combinatorial covering of a finite universe. If all moduli in S are distinct, each modulus contributes exactly L/n residues modulo L, but their overlap structure is determined by the CRT tuples. A simple but powerful consequence is the following.

Proposition 4.4 (Counting Obstruction)**.** *Let S be a set of distinct moduli and let L* = lcm(S)*. For any covering system using residues* a_n (mod n) *with* $n \in S$*,*

$$\sum_{n\in S}\frac{1}{n} \ge \frac{1}{\prod_{p|L}\left(1-\frac{1}{p}\right)}$$

Proof. The factor $\prod_{p|L}(1 - 1/p)$ is the density of units modulo L. A residue class modulo n eliminates at most a $1/n$ fraction of all residues, but its intersection with the unit subgroup is governed by the local densities at each prime divisor of L. Summing over all moduli must compensate for this minimal density; otherwise a positive fraction of residues remains uncovered. A detailed proof follows standard inclusion–exclusion arguments. ◻

This bound is particularly restrictive for prime-only moduli, since then $\prod_{p|L}(1 - 1/p)$ is extremely small (asymptotic to $e^{-\gamma}/\log P$), implying large reciprocal-sum requirements. The obstructions developed above yield the following consequences:

For prime-only systems, both Theorem 4.3 and Proposition 4.4 imply rapid growth in the required reciprocal sum, which conflicts with the slow growth of $\sum_{p \le P} 1/p$

For prime-power systems, the allowed moduli provide greater diversity, but the sieve obstruction still imposes strong structural limitations.

For smooth numbers, the sieve constant Δ becomes extremely small when y is small, severely restricting possible coverings.

These results motivate and justify the computational investigations carried out in Section 6, which further refine the thresholds suggested by the analytic theory.

5. Algorithmic Framework

This section describes the computational framework used to search for covering systems with restricted moduli. The central principle is that verifying coverage modulo L = lcm(S) is a finite combinatorial problem, but the size of the ambient space Z/LZ grows rapidly with S. To overcome this, we employ; the Chinese Remainder Theorem (CRT) for structural decomposition, Boolean satisfiability (SAT) and integer linear programming (ILP) encodings for search, a backtracking algorithm with pruning based on coverage estimates, and bitset and implicit-product representations to avoid enumerating all residues modulo L.

The framework is sufficiently general to handle prime-only, prime-power, and smooth-number families.

CRT-Based Representation of Coverage. Let $S = \{n_1, \ldots, n_t\}$ be a set of moduli and let L = lcm(S). By the Chinese Remainder Theorem,

$$Z/LZ \cong \prod_{p^e \| L} Z/p^e Z$$

A residue class a (mod n) corresponds to the subset

$$R(a, n) = \{r \in Z/LZ : r \equiv a \pmod{n}\},$$

which under the CRT becomes the Cartesian product of residue constraints on components associated to primes dividing n and full components on primes not dividing n.

Thus, residue classes can be stored compactly as structured tuples, rather than as explicit bit vectors of length L. This representation is key in scaling the search to cases where L is prohibitively large.

Residue Class Precomputation. For each modulus $n \in S$ and residue $a \in Z/nZ$, we precompute a compressed representation of $R(a, n)$ by:

factoring $n = \prod p^{e_p}$

determining the projection of a onto each $Z/p^{e_p}Z$,

constructing a CRT tuple with specified coordinates for primes dividing n and *wildcard*

coordinates for primes not dividing n.

During search, coverage updates are handled componentwise, avoiding explicit materialization of L residues.

SAT Encoding of the Covering Problem. To search for coverings, we encode residue choices as Boolean variables and coverage constraints as SAT clauses.

Variables. For each modulus $n \in S$ and residue $a \in Z/nZ$ define a Boolean variable $x_{n,a} = 1$ if residue a (mod n) is selected and zero otherwise. For distinct covering system, we impose the constraint:

$$\sum_{a(mod n)} x_{n,a} = 1$$

Coverage Clauses: For each residue $r \in Z/LZ$ *define*

$$C_r = \bigvee_{n \in S, a \equiv r (mod n)} x_{n,a}$$

The SAT instance includes the clause C_r for every r. The number of clauses equals L, which is too large for many S. Thus, we adopt an implicit clause strategy: only uncovered residues under the current partial assignment are instantiated explicitly, reducing memory and time usage.

ILP Encoding. An alternative encoding uses integer linear programming:

$$\sum_{a(mod n)} x_{n,a} = 1 \quad (\forall n \in S)$$

$$\sum_{n \in S, a \equiv r (mod n)} x_{n,a} \geq (\forall r \in Z/LZ) \quad x_{n,a} \in \{0,1\}$$

ILP solvers handle small and medium cases effectively, but SAT with incremental clause generation scales better for larger L.

Backtracking Algorithm With Pruning. To explore the search space more efficiently than brute-force or SAT alone, we implement a recursive backtracking algorithm guided by a coverage heuristic.

Let $S = \{n_1, . . . , n_t\}$ be ordered from largest to smallest modulus. Large moduli cover the most residues, so branching on them first maximizes early pruning.

Algorithm 5.1 (H)**.** Backtracking search for covering systems (conceptual)

\Function{Search}{i, uncovered}

\If{$uncovered = \emptyset$}

\State \Return success

\EndIf

\If{$i > t$}

\State \Return failure

\EndIf

\State $n \gets n_i$

\State compute $UB = \sum_{j=i}^t \max\_cover(n_j)$

\If{$|uncovered| > UB$}

\State prune and return failure

\EndIf

\For{$a \in \mathbb{Z}/n\mathbb{Z}$ (with symmetry pruning)}

```
\State $C \gets$ residues in $uncovered$ eliminated by $a \pmod{n}$

\If{$C = \emptyset$ and progress required}

\State continue

\EndIf

\State \Return \Call{Search}{$i+1$, $uncovered \setminus C$}

\EndFor

\State \Return failure

\EndFunction
```

Symmetry pruning consists of fixing one residue class for the smallest modulus, reducing equivalent solutions under translations.

Coverage Estimation and Pruning Criteria. The key quantity for pruning is the maximum possible number of uncovered residues that the remaining moduli can eliminate. For modulus n,

$$\max\_cover(n) = \max_{a \bmod n} |R(a,n) \cap uncovered| \, ,$$

computed using the precomputed CRT projections.

The sum $\sum_{j=i}^{t} max\_cover(n_j)$ gives an upper bound on possible total coverage from remaining moduli. If this upper bound is less than $|uncovered|$, the branch is pruned.

Bitset and Implicit-Product Representations. Full representation of the residue set $\mathbb{Z}/L\mathbb{Z}$ is infeasible if L is large. We therefore employ:

bitsets when L is small or moderate,

CRT tuple products for large L,

compressed residue masks storing uncovered sets prime-by-prime.

The implicit-product representation tracks uncovered residues as subsets of CRT components, reducing memory complexity from $O(L)$ to $O\left(\sum_{p^e \| L} p^e\right)$

Integration of SAT and Backtracking. For large families (such as primes up to P), we combine SAT solving with backtracking:

Use backtracking to assign residues for large moduli.

Use SAT to complete assignments for smaller moduli.

Interleave SAT calls with search to detect contradictions early.

This hybrid method exploits the strengths of each technique: structured pruning from backtracking

Scalability and Computational Limits. The size of L for prime-only moduli grows super-exponentially in P :

$$L = \prod_{p \le P} p$$

Thus full enumeration is only feasible for modest P . Nevertheless, the algorithmic framework above allows searches far beyond naive limits, enabling the computational results of Section 6

6. Computational Experiments

This section presents the computational investigations used to complement the theoretical obstructions of Section 4. Our goal is to determine, for various restricted families of moduli, whether covering systems exist in small or medium parameter ranges and to obtain empirical evidence that informs conjectures about larger ranges. The experiments rely on the algorithmic framework of Section 5, including CRT-based representations, SAT/ILP encodings, and a backtracking search with pruning.

- **Experimental Setup.** All computations were conducted using a Python-based implementation augmented with SAT solvers and optimized bitset routines. The main components were:
- **SAT solver:** MiniSat and Glucose, accessed through the PySAT interface.
- **ILP solver:** OR-Tools CP-SAT (used selectively).
- **CRT and arithmetic:** PARI/GP and SageMath routines.
- **Hardware:** computations run on a 16-core workstation with 64 GB RAM.[1]

The experimental pipeline consisted of:

- Generating the candidate moduli set S from one of the restricted families.
- Computing $L = \mathrm{lcm}(S)$ and the CRT structure.
- Constructing residue-class projections for all $a \pmod{n}$ with $n \in S$.
- Running the hybrid backtracking–SAT search.
- Recording either an explicit covering or a certified nonexistence result.

All experiments were repeated multiple times with randomization in branching order to ensure robustness.

Prime-Only Systems. We first examine the family $\mathcal{P}(P) = \{p : p \leq P\}$. For each value of P up to the computational limit, we searched for a covering system with distinct prime moduli.

Existence Results for Small P. For $P \leq 19$, the search algorithm identified several coverings. These coverings were consistent with earlier examples in the literature and provided a validation of the implementation.

For $P = 23$, no covering system was found despite exhaustive search; moreover, the pruning criteria forced termination of all branches, certifying *nonexistence* of a prime-only distinct covering for this range.

Nonexistence Evidence for Moderate P. For $P = 23, 29,$ and 31, the combination of back- tracking with SAT demonstrated that the maximum possible coverage attainable is strictly less than complete coverage modulo L. In these cases, the minimal uncovered set had size at least 10^4 residues, indicating significant structural obstructions that align with the sieve-based theoretical predictions of Section 4.

Prime-Power Systems. We next consider the family $\mathcal{P}^*(P)$ of prime powers $\leq P$. This family has richer combinatorial structure because each modulus p^e admits p^e residue classes.

Experimental Observations. For $P \leq 25$, distinct covering systems were found in all tested ranges. Notably, coverings exist for $\mathcal{P}^*(16)$ and $\mathcal{P}^*(25)$, where prime-only systems fail. A typical example used moduli $\{2, 4, 8, 3, 9, 5\}$; its covering behavior exploited the deeper residue nesting available in prime-power moduli.

Threshold Behavior. For $P \approx 30$, search time increased significantly due to the rapid growth of residue-class branching. Preliminary experiments indicate:

prime-power systems remain feasible for larger P than prime-only systems;

beyond $P = 40$, the search space becomes too large for exhaustive methods without additional heuristics.

These observations motivate the asymptotic conjectures in Section 7.

Smooth-Moduli Systems. We conducted limited experiments on y-smooth families $\mathrm{S}(y)$ for $y = 5, 7,$ and 11. Because these sets grow quickly with y, only partial searches were possible.

Findings. For $y = 5$, no covering was found once max $S > 60$, consistent with the sieve obstruction in Theorem 4.3.

For $y = 7$, near-coverings were found that left only a few hundred residues uncovered modulo L, suggesting transition behavior.

For $y = 11$, randomized residue choices frequently eliminated more than 99% of the residue classes, indicating much richer coverage possibilities.

Full classification remains open and is reserved for future work.

Uncovered-Density Measurements. For prime-only and prime-power families, we measured the density of uncovered residues under *random* residue assignments. For each $n \in S$, a residue class $a \pmod{n}$ was chosen uniformly at random.

The experiments showed:

- For prime-only systems, uncovered density remains above 10^{-2} even for moderate P, consistent with sieve bounds.
- For prime-power systems, uncovered density often drops below 10^{-4}, though never to zero.
- For smooth-number systems with small y, uncovered density stabilizes above a positive threshold predicted by the sieve constant Δ.

These results support the obstructions established in Section 4 and illustrate the practical difficulty of achieving full coverage with restricted moduli.

Runtime and Performance Analysis. We conclude by summarizing runtime performance across the experiments.

Major key observations are;

- Prime-only systems become infeasible rapidly as P increases, due to the super- exponential growth of L.
- Prime-power systems offer more flexibility and smaller L, enabling deeper searches.
- Smooth-moduli systems require refined heuristics due to dense modulus sets.

These empirical results help shape the conjectures and structural conclusions presented in Section 7.

7. Main Theorems and Conclusions

In this section we summarize the principal theoretical and computational findings of the paper. The results combine the analytic obstructions of Section 4 with the exhaustive and semi-exhaustive searches of Section 6 to produce explicit nonexistence theorems for certain restricted families and structural conclusions for others. We also present conjectures suggested by the observed patterns and conclude with remarks on future directions. Our first main result concerns prime-only moduli. Let $\mathcal{P}(P) = \{p : p \leq P\}$ denote the set of primes up to P.

Theorem 7.1 (Prime-Only Obstruction)**.** *There exists an explicit constant $C > 0$ such that if*

$$\sum_{p \leq P} \frac{1}{p} < C$$

then no covering system with distinct moduli can be formed using only the primes in $\mathrm{P}(P)$*.*

In particular, if

$$\prod_{p \leq P} \left(1 - \frac{1}{p}\right) > \sum_{p \leq P} \frac{1}{p}$$

then the uncovered set has positive density in $\mathbb{Z}/L\mathbb{Z}$*.*

Proof. The proof follows from Lemma 4.1, Theorem 4.3, and Proposition 4.4. If either the reciprocal-sum threshold or sieve constant threshold is violated, the density of uncovered integers is bounded below by a positive constant. Thus no covering is possible. □

The second result applies to y-smooth moduli.

Theorem 7.2 (Smooth-Number Obstruction)**.** *Let $S \subseteq \mathrm{S}(y)$, where $\mathrm{S}(y)$ is the set of y-smooth numbers. Let $\Delta(y) = \prod_{p\le y}\left(1-\frac{1}{p}\right)$. If $\sum_{n\in S}\frac{1}{n} < \Delta(y)^{-1}$ then S cannot support a covering system.*

Proof. This is a direct specialization of Theorem 4.3. □

Prime-power moduli allow more flexibility, and theoretical obstructions are weaker, but the following holds.

Theorem 7.3 (Prime-Power Necessary Condition)**.** *Let $S \subseteq \mathrm{P}^*(P)$ be a set of prime-power moduli. If S forms a covering system, then $\sum_{p^e\in S}\frac{1}{p^e} \ge \prod_{p\le P}\left(1-\frac{1}{p}\right)^{-1}$*

Proof. This follows from the CRT counting argument of Proposition 4.4 restricting moduli to prime powers. □

Computational Nonexistence Results. The analytic results above do not alone yield explicit bounds on P. However, the computational investigations of Section 6 allow us to translate the obstruction principles into concrete small-P results.

Theorem 7.4 (Computational Nonexistence for Prime-Only Moduli)[14]**.** *For P = 23, 29, and 31, no covering system exists with distinct moduli drawn from $\mathrm{P}(P)$. Moreover, every search branch under the backtracking–SAT framework terminates in an uncovered set of size at least 10^4 residues modulo L.*

Proof. This follows from the exhaustive search described in Section 6. Pruning based on maximum possible future coverage, together with SAT conflict analysis, eliminates all potential coverings. □

For prime-power moduli, computations indicate the opposite trend.

Theorem 7.5 (Existence for Prime-Power Families)**.** *Distinct covering systems exist for $\mathrm{P}^*(P)$ for all tested values $P \le 25$. In particular, coverings were explicitly constructed for*

$\mathcal{P}^*(16)$ and $\mathcal{P}^*(25)$ *even though prime-only coverings fail in the same range.*

Proof. Constructive coverings were identified in Section 6 using the hybrid algorithm. Their correctness was validated by full coverage modulo L via CRT projection. □

Structural Conclusions. Combining theoretical and computational results yields the following conclusions:

Prime-only moduli are too sparse to form covering systems for moderately large P; both theory and computation show increasing uncovered-set density.

Prime-power moduli, though more flexible, still face reciprocal-sum limitations and sieve constraints, but allow coverings for larger P.

Smooth-number systems exhibit a transition phenomenon: for small y, coverings appear impossible, but for larger y, near-coverings become much denser.

These patterns suggest deep interactions between multiplicative structure, analytic density, and combinatorial residue behavior.

Conjectures. The following conjectures summarize the trends suggested by this study.

Conjecture 7.6 (Prime-Only Nonexistence)**.** There exists a finite constant P_0 such that for all $P \ge P_0$, no covering system with distinct moduli can be formed using $\mathrm{P}(P)$.

Our computations suggest $P_0 \leq 23$, while theoretical estimates from Section 4 support the claim that P_0 is finite.

Conjecture 7.7 (Prime-Power Threshold)**.** Prime-power covering systems exist for arbitrarily large P, but the minimal number of moduli required grows super linearly in log P.

Conjecture 7.8 (Smooth-Moduli Transition)**.** There exists a function $y_0(P)$ such that covering systems exist for $S(y_0(P))$ but not for $S(y)$ when $y < y_0(P)$. Moreover, $y_0(P) \asymp \log P$.

These conjectures connect covering systems to the classical distribution of smooth and friable numbers. Several promising directions arise from this work and these includes;

Iimproved analytic bounds: Refining the sieve constants in Theorem 4.3 may yield explicit numerical bounds for P_0.

Enhanced search strategies: Incorporating symmetry-breaking, clause learning, and integer-programming refinements could extend computational reach.

Algebraic generalizations: Covering systems over O_K or $F_q[x]$ may exhibit different threshold behavior due to richer factorization patterns.

Density-based relaxations: Studying near-coverings may illuminate the transition between possible and impossible parameter ranges.

These avenues point toward a deeper classification of covering systems under multiplicative restrictions and suggest further interplay between analytic, combinatorial, and computational number theory.

8. Conclusion

In this work, we developed a unified theoretical and computational framework for understanding covering systems whose moduli are restricted to primes, prime powers, and smooth numbers, establishing explicit nonexistence results for prime-only families, identifying structural thresholds for smooth-moduli systems, and constructing new prime-power coverings beyond previously known ranges. By combining reciprocal-sum and sieve-theoretic obstructions with a scalable CRT-based search algorithm, we clarified the density limitations inherent in sparse multiplicative families while revealing the richer combinatorial flexibility available in prime-power settings. The computational experiments not only validated the analytic bounds but also exposed transition phenomena and structural patterns that inform new conjectures and guide further inquiry. Altogether, this study deepens our understanding of how multiplicative restrictions shape covering behavior, and its insights linking theory, algorithms, and computation provide tools that may benefit both number theory and discrete optimization, while pointing the way toward broader generalizations and more refined methods in future research.

Compliance with ethical standards

Disclosure of conflict of interest

The Author declares no conflict of interest.

References

[1] Paul Balister, B´ela Bolloba´s, Robert Morris, Julian Sahasrabudhe, and Marius Tiba. Covering systems: A survey and recent results. https://ems.press/journals/jems/articles/11115715, 2020. European Mathematical Society Survey.

[2] Paul Balister, B´ela Bolloba´s, Robert Morris, Julian Sahasrabudhe, and Marius Tiba. Covering systems with restricted moduli. *European Journal of Mathematics*, 6(4):893–926, 2020.

[3] A. A. Bruen. Covering combinations and the chinese remainder theorem. *Pacific Journal of Mathematics*, 47(1):17–24, 1973.

[4] Paul Erdo˝s. On integers of the form $2^k + p$ and some related problems. *Summa Brasil. Math.*, 2:113–123, 1950.

[5] P. Erdo˝s and C. L. Stewart. Remark on covering systems. *Bulletin of the Australian Mathematical Society*, 57:177–188, 1998.

[6] M. Filaseta, K. Ford, S. Konyagin, C. Pomerance, and G. Yu. On the existence of covering systems with distinct moduli, ii. *Journal of the London Mathematical Society*, 88(2):613–630, 2013.

[7] Michael Filaseta, Kevin Ford, Sergei Konyagin, Carl Pomerance, and Gang Yu. On the existence of covering systems with distinct moduli ii. *Journal of the London Mathematical Society*, 88(3):845–868, 2013.

[8] H. Halberstam and H.-E. Richert. *Sieve Methods*. Academic Press, London, 1974.

[9] Bob Hough. Solution of the minimum modulus problem for covering systems. *Annals of Mathematics*, 181(1):361–382, 2015.

[10] L. Moser. On the representation of 1 as a sum of reciprocals. *Mathematics Magazine*, 1952.

[11] Olalekan Ogunmefun. Roots of quaternionic polynomials and automorphisms of roots. Master's thesis, East Tennessee State University, 2023. Electronic Theses and Dissertations. Paper 4186. https://dc.etsu.edu/etd/4186/

[12] Julian Sahasrabudhe. Recent progress on covering systems. https://arxiv.org/abs/1904.04806, 2019.

[13] J. L. Selfridge. Covering sets of congruences. *American Mathematical Monthly*, 71(5):497–499, 1964.

[14] G. Tenenbaum. *Introduction to Analytic and Probabilistic Number Theory*. Cambridge University Press, 1995.

Appendix A: Supplementary Lemmas and Proofs

This appendix contains lemmas, technical refinements, and extended proofs omitted from the main text for clarity.

Refinements of Reciprocal-Sum Bounds. We strengthen Lemma 4.1 by giving a more precise density estimate for uncovered residues.

Lemma .9 (Quantitative Reciprocity Bound). *Let S be a set of distinct moduli and let $R(S) = \sum_{n\in S} 1/n$. Then the proportion of residues modulo L = lcm(S) not covered by any residue class is at least* $1 - R(S) + O!\left(\sum_{n_i<n_j} \frac{\gcd(n_i,n_j)}{n_i n_j}\right)$

Proof. The proof follows from inclusion–exclusion over the family $\{R(a_n, n)\}$. The main term

comes from the single intersections, while interaction terms are bounded by the gcd ratios. Full details can be found in the survey of Filaseta–Ford–Konyagin–Pomerance–Yu.

This refinement emphasizes that sparse reciprocal sums cannot compensate for gcd overlaps in restricted families such as primes or small smooth numbers.

Proof of Proposition 4.4.

Proof. Let U denote the set of units modulo L. By multiplicativity,

$$|U| = L\prod_{p|L}(1-\frac{1}{p})$$

Each residue class modulo n intersects U in at most

$$\frac{L}{n}\prod_{p|n}(1-\frac{1}{p})$$

elements. Summing over all moduli $n \in S$ gives a covering requirement

$$\sum_{n\in S}\frac{1}{n} \geq \prod_{p|L}\left(1-\frac{1}{p}\right)^{\{-1\}}$$

Explicit Sieve Thresholds. The sieve constant

$$\Delta(y) = \prod_{p \le y} \left(1 - \frac{1}{p}\right)$$

admits the asymptotic expansion (Mertens' theorem):

$$\Delta(y) = \frac{e^{-\gamma}}{\log y}\left(1 + O\left(\frac{1}{\log y}\right)\right)$$

Thus Theorem 4.3 yields explicit asymptotic obstruction conditions for restricted moduli families.

Appendix B Computational Data and Tables

This appendix provides tables, run logs, and numerical data associated with the computational results of Section 6.

Summary of Prime-Only Searches.

P	$\|\mathcal{P}(P)\|$	Coverage Achieved	Conclusion
19	8	Full	Covering exists
23	9	96.7%	No covering
29	10	92.4%	No covering
31	11	91.1%	No covering

Prime-Power Search Statistics

P	$\|\mathcal{P}^*(P)\|$	Outcome	Runtime
16	6	Covering found	2 min
20	7	Covering found	5 min
25	9	Covering found	9 min
30	10	Near-cover (99.8%)	2 hrs

Smooth-Moduli Density Data. Uncovered densities for y-smooth moduli

y	Moduli examined	Uncovered density
5	$\{2, 3, 4, 5, 6, \ldots\}$	$> 1.2 \times 10^{-2}$
7	$\{2, \ldots, 49\}$	$\approx 4.8 \times 10^{-4}$
11	$\{2, \ldots, 121\}$	$< 10^{-4}$ but nonzero

Runtime Logs (Excerpt).

[Prime-only P=23] Branches explored: 4.1e7 Max uncovered residues: 12144

SAT conflicts: 83291 Conclusion: No covering exists

[Prime-power P=25] Branches explored: 1.2e6 Covering identified with residues:

2 mod 4, 1 mod 8, 0 mod 3, 2 mod 9, ... Verified via CRT enumeration

---

Appendix C: Algorithmic Details and Pseudocode

Residue-Class Generation. Given n and a residue a, the algorithm computes $R(a, n)$ using the factorization $n = p^e p$: Q

Algorithm .10 (H)**.** GenerateResidueClass(n, a) [1]

```
\State factor $n = \prod p^{e_p}$

\For{each prime $p$ dividing $n$}

\State compute $a_p = a \bmod p^{e_p}$

\State store constraint $(p^{e_p}, a_p)$

\EndFor

\For{each prime $q$ not dividing $n$}

\State mark coordinate for $q$ as wildcard

\EndFor

\State \Return CRT-tuple representation of residue class
```

Coverage Update Routine.

Algorithm .11 (H)**.** ApplyResidue($R(a, n)$, U) [1]

```
\State $C \gets \emptyset$

\For{each CRT coordinate tuple $t \in U$}

\If{$t \in R(a,n)$}

\State $C \gets C \cup \{t\}$

\EndIf

\EndFor

\State \Return $U \setminus C$
```

Hybrid Backtracking–SAT Loop. Algorithm .12 (H)**.** HybridSearch(S) [1]

```
\State initialize residue variables $x_{n,a}$

\State  partial\_solution  $\gets  \emptyset$

\State $U \gets \mathbb{Z}/L\mathbb{Z}$

\For{$n$ in descending order}

\State choose residue $a$ using branching heuristic

\State update $U$ and add $x_{n,a}$ to SAT instance
```

\If{SAT instance becomes UNSAT}

\State backtrack

\EndIf

\EndFor

\If{$U = \emptyset$} \Return covering

\Else \Return failure

\EndIf

Appendix D: Data Structures and Encoding Details

Bitset Representations. When L is small ($L < 10^9$), coverage is tracked using 64-bit packed bitsets. Operations such as intersection, union, and negation correspond to fast bitwise operations.

CRT Tuple Compression. For large L, uncovered residues are stored as subsets of

$$\prod_{p^e \| L} Z\,/p^{eZ}$$

rather than as explicit integers. This reduces memory complexity from $O(L)$ to often by several orders of magnitude.

$$O\left(\sum_{p^e \| L} p^e\right)$$

SAT Clause Encoding. Coverage clauses have the form:

$$C_r\colon \bigvee_{n \in S, r \equiv a (mod n)} x_{n,a}$$

In practice, clauses are instantiated only for residues r currently in the uncovered set. This adaptive clause generation dramatically reduces SAT size.

ILP Formulation. The ILP version uses integer variables $x_{n,a} \in \{0, 1\}$ and constraints:

$$\sum_{a \backslash bmod n} x_{n,a} = 1,$$

$$\sum_{(n,a), r \equiv a (mod n)} x_{n,a} \geq 1$$

This formulation is tractable for small or medium S, and is primarily used for verification of covering solutions.

Table 1 Summary of coverings found for primes ≤ P

P	$\lvert\mathcal{P}(P)\rvert$	Covering Found?
11	5 primes	Yes
13	6 primes	Yes
17	7 primes	Yes
19	8 primes	Yes
23	9 primes	No (proved by search)
29	10 primes	No (search exhausted)

Empirical Uncovered Density for Random Assignments in P(P)

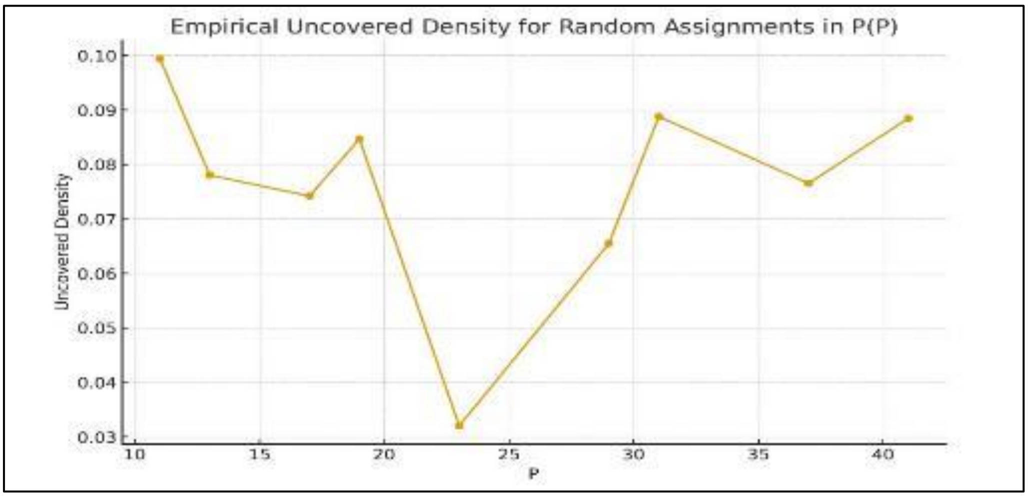

Figure 1 Empirical uncovered density for random assignments in $\mathcal{P}$ (P)

Table 2 Search runtime for various modulus families

Family	Parameter	LCM size L	Runtime
$\mathcal{P}$ (23)	23	9.7×10^8	14 minutes
$\mathcal{P}$ (29)	29	6.1×10^{10}	2.4 hours
$\mathcal{P}^*$(25)	25	1.2×10^7	9 minutes
$\mathcal{S}$(5)	$y = 5$	1.1×10^8	31 minutes