



(RESEARCH ARTICLE)

Golden-Thompson inequality for matrix powers of matrix

K. K. W. A. S. Kumara ^{1,*} and G. Nandasena ²

¹ *Department of Mathematics, Faculty of Applied Science, University of Sri Jayewardenepura, Gangodawilla, Nugegoda 10250, Sri Lanka.*

² *Department of Mathematics and Philosophy of Engineering, Faculty of Engineering Technology, The Open University of Sri Lanka, Nawala, Nugegoda, 10250, Sri Lanka.*

World Journal of Advanced Engineering Technology and Sciences, 2026, 20(01), 094–097

Publication history: Received on 30 May 2026; revised on 06 July 2026; accepted on 08 July 2026

Article DOI: <https://doi.org/10.30574/wjaets.2026.20.1.0350>

Abstract

The paper provides a concise exposition of the mathematical structure related to the Golden Thomson inequality, focusing specifically on matrix exponentials, matrix logarithms, trace functions, and Hermitian matrices within matrix analysis. In this study, we establish the Golden-Thomson inequality for matrix powers of a positive definite Hermitian matrix.

Keywords: Matrix exponentials; Matrix logarithms; Trace function; Hermitian matrices; Golden-Thomson inequality

1. Introduction

This paper focuses on the theory of matrix exponentials and related operator inequalities. We begin by introducing the definition of the matrix exponential along with its basic properties. We also discuss the matrix logarithm and summarize several of its fundamental characteristics.

The matrix exponential is defined via its Taylor series expansion, which provides the natural frameworks for studying its properties. In particular, we examine the exponential identity $e^{A+B} = e^A e^B$. For commuting matrices A and B , this identity holds, since the condition $AB = BA$ allows the terms in the power series expansion to be rearranged consistently. However, this identity is not valid in general for non-commuting matrices, and it fails whenever A and B are not commutative.

Next, we develop the concept of matrix powers for a given matrix and present their main algebraic properties. After indicating these preliminaries, we introduce the Golden-Thompson trace inequality, an important result in matrix analysis and mathematical physics. Finally, we extend the discussion by proving a version of the Golden-Thompson inequality for matrix powers of a positive definite Hermitian matrix.

2. Matrix exponential and Logarithm

In this section, we present the preliminary concepts and results required for the main theorem of this paper. We begin by recalling the definitions and basic properties of the matrix exponential and the matrix logarithm, with particular attention to Hermitian and positive definite matrices. We then state the Golden-Thompson inequality, which plays a central role in the proof of our main result. These concepts provide the theoretical foundation for establishing the Golden-Thompson inequality for matrix powers of positive definite Hermitian matrices.

* Corresponding author: K. K. W. A. S. Kumara.

2.1. The Exponential of a Matrix

The matrix exponential is a fundamental concept that generalizes the classical exponential function from scalars to square matrices.

2.1.1. Definition

Let X be an $n \times n$ matrix, i.e., $X \in M_n(K)$, where $K = \mathbb{R}$ or $K = \mathbb{C}$. We define the exponential of X , denoted e^X or $\exp(X)$ by the usual power series:

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!},$$

where $X^0 = I$ and I is the identity matrix of order n .

2.1.2. Proposition ([1], Proposition 2.2)

Let $Y \in M_n(\mathbb{K})$. Then,

- (i) $(e^X)^* = e^{X^*}$
- (ii) If $XY = YX$, then $e^{X+Y} = e^X e^Y = e^Y e^X$.

2.2. The Matrix Logarithm ([2])

The matrix logarithm is the inverse operation of the matrix exponential. It extends the scalar natural logarithm to square matrices.

2.2.1. Definition

The logarithm of a matrix is defined as the inverse of the matrix exponential, as far as possible. Let:

$$A \in N_{M_n}(I, 1) = \{A \in M_n \mid \|A - I\| < 1\}.$$

Then, for $A \in N_{M_n}(I, 1)$ the logarithm is given by the power series.

$$\ln A = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (A - I)^n.$$

2.3. Trace function

The trace function is a fundamental operation in linear algebra that assign the scalar value to a square matrix.

2.3.1. Definition

Let $A \in \mathbb{R}^{n \times n}$ (or $\mathbb{C}^{n \times n}$). The trace function is defined as the following map.

$\text{Tr}: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ (or $\mathbb{C}$) given by:

$$\text{Tr}(A) = \sum_{i=1}^n a_{ii},$$

where a_{ii} denotes the diagonal entries of the square matrix A .

3. Golden-Thompson Inequality

The Golden-Thompson inequality is a fundamental result in matrix analysis, particularly in the study of Hermitian matrices and matrix exponentials.

3.1. Proposition (Golden-Thompson Inequality) ([3])

If A and B are Hermitian matrices, then $\text{Tr}(e^{A+B}) \leq \text{Tr}(e^A e^B)$.

4. Matrix Powers of a matrix

In this section, we introduce the notion of matrix powers for positive definite Hermitian matrices and present the fundamental properties needed in the proof of the main theorem. In particular, we define matrix powers in terms of the matrix exponential and logarithm, and discuss their basic algebraic and analytic properties. These results will be used in the next section to establish the Golden–Thompson inequality for matrix powers.

4.1. Definition ([4])

Let $A \in N_{M_n}(I, 1)$ and $B \in M_n$. The matrix power of A to B is defined as:

$$A^B := e^{B \ln A}.$$

4.2. Proposition ([2])

Let $A \in N_{M_n}(I, 1)$ and $B, C \in M_n$, $\|B\| < 1$ and $\|C\| < 1$. Then,

- (i) $A^{0_{n \times n}} = I_{n \times n}$.
- (ii) If $BC = CB$, then $A^{B+C} = A^B A^C$.

5. Golden-Thompson Inequality for matrix powers of matrix

In this section, we establish the Golden-Thompson inequality for matrix powers of a positive definite Hermitian matrix, stated as the following theorem.

5.1. Theorem

Let $A \in N_{M_n}(I, 1)$ and $B, C \in M_n$. If A be a positive definite Hermitian matrix and, B and C be Hermitian matrices, then $\text{Tr}(A^{B+C}) \leq \text{Tr}(A^B A^C)$.

Proof:

Suppose A is a positive definite Hermitian matrix and B and C be Hermitian matrices.

Then, $\ln A$ is Hermitian.

Therefore, $B \ln A$ and $C \ln A$ are Hermitian, and hence, $B \ln A + C \ln A$ is Hermitian.

Thus, by the definition $A^B := e^{B \ln A}$ and proposition 3.1

we have

$$\begin{aligned} & \text{Tr}(A^{B+C}) \\ &= \text{Tr}(e^{(B+C) \ln A}) \\ &= \text{Tr}(e^{B \ln A + C \ln A}) \\ &\leq \text{Tr}(e^{B \ln A} e^{C \ln A}) \\ &= \text{Tr}(A^B A^C). \end{aligned}$$

This completes the proof. ■

6. Conclusions

In this paper, we generalized the Golden–Thompson type inequality for matrix powers of a positive definite Hermitian matrix. Specifically, for a positive definite Hermitian matrix A and Hermitian matrices B and C , we proved that $\text{Tr}(A^{B+C}) \leq \text{Tr}(A^B A^C)$.

The proof was obtained by expressing matrix powers in exponential form through the definition $A^B = e^{B \ln A}$, where $\ln A$ is Hermitian, since A is positive definite Hermitian. This representation allowed us to apply the classical Golden–Thompson trace inequality to the Hermitian matrices $B \ln A$ and $C \ln A$, thereby deriving the required result.

Thus, the theorem provides a natural extension of the classical Golden–Thompson inequality from matrix exponentials to matrix powers. This highlights the usefulness of the logarithmic representation of matrix powers and the close connection between trace inequalities and matrix functions. The result may also serve as a basis for further investigations into analogous inequalities for other classes of matrix functions and operators.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

References

- [1] Brian C. Hall, Lie Groups, Lie Algebras, and Representations, An Elementary Introduction. Springer, 2003.
- [2] Kumara KKWAS, Nandasena G, Algebraic Structure of Matrix Powers of Matrix World Journal of Advanced Engineering Technology and Sciences, 2025, 16(03), 010-014
- [3] Sidney Golden's, "Lower Bounds for the Helmholtz Function" was published in Physical Review, volume 137, issue 4B, pages B1127–B1128, on February 22, 1965.
- [4] Kumara KKWAS, Computing the Matrix Powers of Matrix International Journal of Advanced Research (IJAR), Vol. 09, 21 May 2021.