World Journal of Advanced Engineering Technology and Sciences
eISSN: 2582-8266
Cross Ref DOI: 10.30574/wjaets
Journal homepage: https://wjaets.com/

(Research Article)

BIAS OF TRIANGULAR SIGNAL AMPLITUDE ESTIMATION USING THREE-PARAMETER SINE FITTING IN THE PRESENCE OF ADDITIVE NOISE

F. Corrêa Alegria *

Instituto de Telecomunicações and Department of Electrical and Computer Engineering, Instituto Superior Técnico, University of Lisbon, Portugal.
* Corresponding Author

ORCID Details
Francisco A. C. Alegria: https://orcid.org/0000-0003-0854-489X

World Journal of Advanced Engineering Technology and Sciences, 2026, 20(02), 253–269

Publication history: Received on 17 July 2026; revised on 23 August 2026; accepted on 25 August 2026

Article DOI: https://doi.org/10.30574/wjaets.2026.20.2.0420

ABSTRACT

Triangular waveforms are widely used as stimulus signals in instrumentation and measurement, and their amplitude is frequently recovered by applying a least-squares sine-fitting algorithm to the signal's Fourier fundamental. This paper examines the accuracy of that approach when the acquired samples are corrupted by additive Gaussian noise. Because the three-parameter sine-fit estimator is non-linear in the amplitude, the recovered value is systematically biased — a behaviour that, for a triangular input, does not appear to have been quantified before. A compact approximate expression is derived for the relative amplitude error in terms of the signal amplitude, the noise standard deviation and the number of acquired samples. It shows that the bias, though never exactly zero for a finite record, diminishes as the number of samples grows and vanishes in the limit, so that the estimator is asymptotically unbiased. The expression is confirmed by both Monte Carlo simulation and measurements on a data-acquisition system, and it allows a confidence interval to be assigned to triangular-amplitude measurements or, when the noise level is known, the bias to be corrected. Only the three-parameter algorithm under coherent sampling is considered.

Keywords: Sine Fitting; Three-Parameter Sine Fitting; Amplitude Estimation; Bias; Additive Noise; Triangular Signal; Analog-To-Digital Converter.

1 INTRODUCTION

Sine fitting is the standard procedure for recovering the parameters of a sinusoidal signal from a set of sampled data points. The algorithm selects the amplitude, phase and offset that minimise the sum of the squared differences between the model and the acquired samples, and it is applied routinely wherever a periodic signal must be characterised accurately.

When the signal frequency is known, the three-parameter variant is preferred: it is non-iterative and returns closed-form estimates of amplitude, initial phase and offset. Amplitude estimation in particular underpins applications as varied as astrophysics [1], computer vision [2], the characterisation of analog-to-digital converters (ADCs) [3], [4] and impedance measurement [5]. The reach of this class of measurement and signal-processing instrumentation extends well beyond electronics: closely related acquisition and estimation techniques support, for instance, geophysical surveying of soil properties [19] and experimental hydraulics, where the three-dimensional flow field at open-channel confluences is reconstructed from densely sampled velocity data [20], [21]. The accuracy with which a signal amplitude can be recovered is therefore a concern common to a broad range of disciplines.

Within converter metrology in particular, sine-fitting and histogram-based procedures are the established tools for quantifying ADC performance, and the estimators they rely on have been characterised in detail — for the gain and offset errors [16], for the random-noise and effective-resolution figures obtained under the IEEE 1057 standard [17], and for the small-amplitude histogram variants used when only part of the input range is exercised [18]. These procedures are codified in the IEEE 1241 standard [4], whose linearity tests are frequently carried out with triangular stimuli of exactly the kind considered in this paper, which makes the accuracy of triangular-amplitude estimation directly relevant to that context.

Many non-ideal effects degrade these estimates, among them additive noise, oscillator phase noise, sampling jitter, quantisation, harmonic and spurious distortion, and frequency error in the stimulus or sampling clock. For a pure sinusoid corrupted by Gaussian additive noise, the precision of the amplitude estimate — and in particular its Cramér–Rao lower bound — has been treated extensively [6]–[10], and the uncertainty of the complete set of sine-fit parameters recovered from digital data has likewise been quantified [15]. The companion question of estimator bias has received far less attention. A bias induced by jitter is noted in [11], but no expression is given for it, and the asymptotic bias of the IEEE 1057 algorithm under jitter is obtained in [12] only in the limit of an infinitely long record. The additive-noise bias of the amplitude recovered from a triangular signal has, to the author's knowledge, not been addressed at all, and it is this gap that the present paper sets out to fill.

The paper therefore analyses the bias of the amplitude estimate obtained when a triangular signal is measured by sine-fitting its fundamental component. The starting point is the Fourier-series description of the waveform.

The Fourier series of a triangular signal is represented by

$$s(t) = \frac{8}{\pi^2}\sum_{k=0}^{\infty}(-1)^k \frac{A_{TRI}\sin(2\pi(2k+1)ft)}{(2k+1)^2}, \tag{1}$$

where A_{TRI} is the amplitude and f is the frequency of the triangular signal. If we consider $k = 0$ we obtain the fundamental of the signal, given by

$$s(t)_{k=0} = \frac{8}{\pi^2} A_{TRI}\sin(2\pi ft). \tag{2}$$

The sinusoidal model can be defined by,

$$z(t) = C + A\cos(\omega t + \varphi), \quad A = \sqrt{{A_I}^2 + {A_Q}^2} \tag{3}$$

where A is the amplitude, ω is the angular frequency, φ is the initial phase, C is the offset, A_I is the in-phase amplitude and A_Q is the in-quadrature amplitude of the sine wave. The model is not linear in the parameters (A, φ and C) and so the estimator it yields is not necessarily unbiased.

Beyond drawing attention to the existence of this bias, the aim is to derive an analytical expression for it — one that can be used to place confidence intervals on the measured amplitude or, when the noise standard deviation is known, to correct the estimate.

As will be shown, the bias is small even for substantial noise levels. This does not diminish the practical relevance of the result: in applications where the noise is comparable to the signal amplitude, or where very high accuracy is demanded, the ability to quantify the bias becomes essential.

2 SINE WAVE FITTING

Consider M data points $z_1, z_2, \ldots, z_M$ given by

$$z_i = C + A\cos(\omega_x t_i + \varphi) \quad (4)$$

where φ is the initial phase and ω_x is the angular frequency ($2\pi f_x$). We consider the phase φ to be a random variable uniformly distributed in an interval with length 2π.

This data is affected by additive voltage white Gaussian noise, d_i, with null mean and standard deviation σ_V:

$$y_i = z_i + d_i = C + A\cos(\omega_x t_i + \varphi) + d_i \quad (5)$$

We wish to estimate the sine wave that best fits, in a least square error sense, to these M points. The estimates of the sine wave are obtained, in a matrix form, with [3]

$$\begin{bmatrix} A_I \\ A_Q \\ C \end{bmatrix} = \left(D^T D\right)^{-1} D^T \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_M \end{bmatrix} \quad (6)$$

with

$$D = \begin{bmatrix} \cos(\omega_a t_1) & \sin(\omega_a t_1) & 1 \\ \cos(\omega_a t_2) & \sin(\omega_a t_2) & 1 \\ \dots & \dots & \dots \\ \cos(\omega_a t_M) & \sin(\omega_a t_M) & 1 \end{bmatrix} \quad (7)$$

and

$$A = \sqrt{A_I^{\ 2} + A_Q^{\ 2}} \quad (8)$$

where ω_a is the angular frequency of the sinusoid we are trying to adjust to the data.

Here we will assume that the number of samples acquired (M) covers exactly an integer number of periods (J) of the sine wave we are trying to fit to the data. This means that the sine wave frequency (f_a), sampling frequency (f_s) and number of samples satisfy

$$\frac{f_a}{f_s} = \frac{J}{M}, \ J \in \quad . \quad (9)$$

Note that J and M should be mutually prime so that the M different samples acquired at M different time instants, correspond to M different sine wave phases. If not, one will have less than M different phases which will increase the uncertainty in the estimation of the sine wave parameters. In the case that J is a multiple of $M/2$, the sampling instants will correspond to only 2 different sine wave phases and matrix $D^T D$ will be singular and hence not invertible (you can not estimate the 3 sine wave parameters with only two data points).

Note that the sampling instants are given by $t_i = i/f_s$. The assumption is reasonable because we can choose whatever values we want for those frequencies and the number of samples. In practice, however, due to instrument inaccuracies, the actual value of those frequencies may not be exactly the values chosen and which satisfy (9)

If the samples cover an integer number of sine wave periods, we have

$$\sum_{i=1}^{M} \cos(\omega_a t_i) = 0, \ \sum_{i=1}^{M} \sin(\omega_a t_i) = 0,$$
$$\sum_{i=1}^{M} \cos(\omega_a t_i)\sin(\omega_a t_i) = 0 \quad (10)$$

and

$$\sum_{i=1}^{M}\cos^2(\omega_a t_i)=\frac{M}{2}\qquad \sum_{i=1}^{M}\sin^2(\omega_a t_i)=\frac{M}{2}\ . \qquad (11)$$

Consequently the inverse of matrix D^TD becomes

$$\left(D^T D\right)^{-1}=\frac{1}{M}\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}. \quad (12)$$

The sine wave parameters can thus be estimated with

$$\begin{bmatrix}A_\mathrm{I}\\ A_\mathrm{Q}\\ C\end{bmatrix}=\begin{bmatrix}\frac{2}{M}\sum_{i=1}^{M}y_i\cos(\omega_a t_i)\\ \frac{2}{M}\sum_{i=1}^{M}y_i\sin(\omega_a t_i)\\ \frac{1}{M}\sum_{i=1}^{M}y_i\end{bmatrix}, \qquad (13)$$

and the square of the sine wave amplitude is given by

$$A^2=A_\mathrm{I}^{\,2}+A_\mathrm{Q}^{\,2}=\frac{4}{M^2}\sum_{i=1}^{M}\sum_{j=1}^{M}y_i y_j\cos\left[\omega_a\left(t_i-t_j\right)\right]\ . \qquad (14)$$

The sine wave amplitude estimator is thus the square root of the value given by (14)

$$A=\sqrt{\frac{4}{M^2}\sum_{i=1}^{M}\sum_{j=1}^{M}y_i y_j\cos\left[\omega_a\left(t_i-t_j\right)\right]}\ . \qquad (15)$$

The approach taken here is to determine the mean (section 3) and the variance (section 4) of the square of the estimated amplitude and then use the results obtained to write an approximate expression for the expected value of the estimated amplitude (section 5).

3 MEAN OF SQUARE ESTIMATED AMPLITUDE

The expected value of the square of the estimated sine wave amplitude is, from (14)

$$\mathrm{E}\left\{A^2\right\}=\frac{4}{M^2}\sum_{i,j}\mathrm{E}\left\{y_i y_j\right\}\cos\left[\omega_a\left(t_i-t_j\right)\right]\ . \qquad (16)$$

To simplify the notation we will abstain, from now on, to indicate the limits of the summations and assume that they span all the integers from 1 to M. In (16)i and j go from 1 to M like the summations in (14)

The expected value of the product of y_i and y_j is, using (5)

$$\mathrm{E}\left\{y_i y_j\right\}=\mathrm{E}\left\{z_i z_j\right\}+\mathrm{E}\left\{d_i d_j\right\}+2\mathrm{E}\left\{z_i d_j\right\}\ . \qquad (17)$$

Since the null mean additive noise d is considered here independent of the stimulus signal,

$$\mathrm{E}\left\{z_i d_j\right\}=\mathrm{E}\left\{z_i\right\}\mathrm{E}\left\{d_j\right\}=0$$. Equation (17)

$$\mathrm{E}\left\{y_i y_j\right\} = \mathrm{E}\left\{z_i z_j\right\} + \mathrm{E}\left\{d_i d_j\right\}. \qquad (18)$$

Considering that the additive noise of two different samples is uncorrelated, we have

$$\mathrm{E}\left\{y_i y_j\right\} = \begin{cases} \mathrm{E}\left\{z_i^2\right\} + \sigma_v^2 \ , \ i = j \\ \mathrm{E}\left\{z_i z_j\right\} \ , \ i \neq j \end{cases}. \qquad (19)$$

It is possible now to use (19)(16)i and j are equal or not. In order to proceed with the derivation we need to have complete summations, that is, summations whose indices span all possible values, and that have in its argument a single expression for all cases of the indices. This can be achieved by splitting the summation in (16)

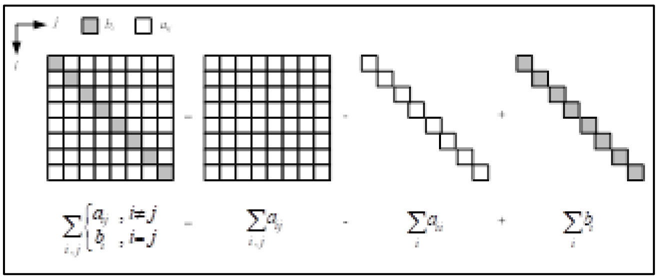

Figure 1: Illustration of a double summation split into three other summations (one double and two single).

Inserting (19)(16)

$$\mathrm{E}\left\{A^2\right\} = \frac{4}{M^2}\sum_{i,j}\mathrm{E}\left\{z_i z_j\right\}\cos\left[\omega_a\left(t_i - t_j\right)\right] - \\ -\frac{4}{M^2}\sum_i \mathrm{E}\left\{z_i z_i\right\} + \frac{4}{M^2}\sum_i\left[\mathrm{E}\left\{z_i^2\right\} + \sigma_v^2\right]. \qquad (20)$$

This can be simplified to

$$\mathrm{E}\left\{A^2\right\} = \frac{4}{M^2}\sum_{i,j}\mathrm{E}\left\{z_i z_j\right\}\cos\left[\omega_a\left(t_i - t_j\right)\right] + \frac{4}{M}\sigma_v^2. \qquad (21)$$

Using (4)

$$\begin{aligned} \mathrm{E}\left\{z_i z_j\right\} &= \mathrm{E}\left\{\begin{matrix}\left[C + A\cos\left(\omega_x t_i + \varphi\right)\right]\times \\ \times\left[C + A\cos\left(\omega_x t_j + \varphi\right)\right]\end{matrix}\right\} = \\ &= C^2 + 2CA \times \mathrm{E}\left\{\cos\left(\omega_x t_i + \varphi\right)\right\} + \\ &+ A^2 \times \mathrm{E}\left\{\cos\left(\omega_x t_i + \varphi\right)\cos\left(\omega_x t_j + \varphi\right)\right\} = \\ &= C^2 + A^2 \times \mathrm{E}\left\{\cos\left(\omega_x t_i + \varphi\right)\cos\left(\omega_x t_j + \varphi\right)\right\}, \qquad (22) \end{aligned}$$

making use of the fact that, since φ is a uniformly distributed random variable between 0 and 2π, $\mathrm{E}\left\{\cos\left(\omega_x t_i + \varphi\right)\right\}$ is null. Transforming the product of two cosine functions into the sum of two cosine functions,

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b), \qquad (23)$$

(22) becomes

$$\mathrm{E}\left\{z_i z_j\right\} = C^2 + A^2 \frac{1}{2}\mathrm{E}\left\{\cos\left(\omega_x t_i + \omega_x t_j + 2\varphi\right)\right\} + \\ + A^2 \frac{1}{2}\mathrm{E}\left\{\cos\left(\omega_x t_i - \omega_x t_j\right)\right\} \quad . \qquad (24)$$

Again, because φ is a uniformly distributed random variable between 0 and 2π, the second term in the second member of (24)

$$\mathrm{E}\left\{z_i z_j\right\} = C^2 + \frac{1}{2}A^2 \cos\left[\omega_x\left(t_i - t_j\right)\right] . \quad (25)$$

Inserting (25)(20)

$$\mathrm{E}\left\{A^2\right\} = \frac{4}{M}\sigma_v^2 + \frac{4}{M^2}C^2 \sum_{i,j}\cos\left[\omega_a\left(t_i - t_j\right)\right] + \\ + \frac{2}{M^2}A^2 \sum_{i,j}\cos\left[\omega_x\left(t_i - t_j\right)\right]\cos\left[\omega_a\left(t_i - t_j\right)\right] \quad . \qquad (26)$$

Considering that the frequency of the sinewave model (ω_x) is known and used as the frequency of the sinewave that is being fitted to the data (ω_a), that is $\omega_x = \omega_a$, and that the data spans an integer number of periods, (26)

$$\mathrm{E}\left\{A^2\right\} = A^2 + \frac{4}{M}\sigma_v^2 \quad . \qquad (27)$$

4 VARIANCE OF THE ESTIMATED SQUARE AMPLITUDE

The variance of the square of the estimated amplitude can be expressed as the difference between two expected values:

$$\sigma_{A^2}^2 = \mathrm{E}\left\{A^4\right\} - \mathrm{E}^2\left\{A^2\right\} . \qquad (28)$$

Using (14)

$$A^4 = \frac{16}{M^4}\sum_{i,j,k,l} y_i y_j y_k y_l \cos\left[\omega_a\left(t_i - t_j\right)\right]\cos\left[\omega_a\left(t_k - t_l\right)\right] \quad .(29)$$

The expected value is thus

$$\mathrm{E}\left\{A^4\right\} = \frac{16}{M^4}\sum_{i,j,k,l}\left\{\begin{array}{l}\mathrm{E}\left\{y_i y_j y_k y_l\right\}\times \\ \times\cos\left[\omega_a\left(t_i - t_j\right)\right]\times \\ \times\cos\left[\omega_a\left(t_k - t_l\right)\right]\end{array}\right\} . \qquad (30)$$

Inserting (16)(30)(28)

$$\sigma_{A^2}^2 = \frac{16}{M^4}\sum_{i,j,k,l}\left\{\begin{array}{l}\mathrm{cov}\left\{y_i y_j, y_k y_l\right\}\times \\ \times\cos\left[\omega_a\left(t_i - t_j\right)\right]\times \\ \times\cos\left[\omega_a\left(t_k - t_l\right)\right]\end{array}\right\} , \qquad (31)$$

where the definition of covariance,

$$\mathrm{cov}\{a,b\} = \mathrm{E}\{ab\} - \mathrm{E}\{a\}\mathrm{E}\{b\} , \qquad (32)$$

was used.

Using (5)

$$\operatorname{cov}\{y_i y_j, y_k y_l\} = \operatorname{cov}\begin{Bmatrix} z_i z_j + d_i d_j + z_i d_j + d_i z_j, \\ z_k z_l + d_k d_l + z_k d_l + d_k z_l \end{Bmatrix}, \qquad (33)$$

which can also be written as

$$\begin{aligned}
&\operatorname{cov}\{y_i y_j, y_k y_l\} = \\
&= \operatorname{cov}\{z_i z_j, z_k z_l\} + \operatorname{cov}\{z_i z_j, d_k d_l\} + \\
&+ \operatorname{cov}\{z_i z_j, z_k d_l\} + \operatorname{cov}\{z_i z_j, d_k z_l\} + \\
&+ \operatorname{cov}\{d_i d_j, z_k z_l\} + \operatorname{cov}\{d_i d_j, d_k d_l\} + \\
&+ \operatorname{cov}\{d_i d_j, z_k d_l\} + \operatorname{cov}\{d_i d_j, d_k z_l\} + \\
&+ \operatorname{cov}\{z_i d_j, z_k z_l\} + \operatorname{cov}\{z_i d_j, d_k d_l\} + \\
&+ \operatorname{cov}\{z_i d_j, z_k d_l\} + \operatorname{cov}\{z_i d_j, d_k z_l\} + \\
&+ \operatorname{cov}\{d_i z_j, z_k z_l\} + \operatorname{cov}\{d_i z_j, d_k d_l\} + \\
&+ \operatorname{cov}\{d_i z_j, z_k d_l\} + \operatorname{cov}\{d_i z_j, d_k z_l\}
\end{aligned} \quad . \qquad (34)$$

Since d is independent on z, every term with 1 or 3 times variable d will be zero. The terms with covariance of the product of two z variables with two d variables is null because the additive noise is considered independent of the signal. The same goes for the term $\operatorname{cov}\{d_i d_j, z_k z_l\}$. Using this we have, from (34)

$$\begin{aligned}
\operatorname{cov}\{y_i y_j, y_k y_l\} &= \operatorname{cov}\{z_i z_j, z_k z_l\} + \operatorname{cov}\{d_i d_j, d_k d_l\} + \\
&+ \operatorname{cov}\{z_i d_j, z_k d_l\} + \operatorname{cov}\{z_i d_j, d_k z_l\} + \\
&+ \operatorname{cov}\{d_i z_j, z_k d_l\} \operatorname{cov}\{d_i z_j, d_k z_l\}
\end{aligned} \quad . \qquad (35)$$

The remaining terms with two z and two d are all equal since multiplication is commutative. We thus have

$$\begin{aligned}
\operatorname{cov}\{y_i y_j, y_k y_l\} &= \operatorname{cov}\{z_i z_j, z_k z_l\} + \\
&+ \operatorname{cov}\{d_i d_j, d_k d_l\} + 4\operatorname{cov}\{z_i d_j, z_k d_l\}
\end{aligned} \quad . \qquad (36)$$

Inserting into (31)

$$\begin{aligned}
\sigma^2_{A^2} &= \frac{16}{M^4} \sum_{i,j,k,l} \operatorname{cov}\{z_i z_j, z_k z_l\} \cos\left[\omega_a (t_i - t_j)\right] \cos\left[\omega_a (t_k - t_l)\right] + \\
&+ \frac{16}{M^4} \sum_{i,j,k,l} \operatorname{cov}\{d_i d_j, d_k d_l\} \cos\left[\omega_a (t_i - t_j)\right] \cos\left[\omega_a (t_k - t_l)\right] + \\
&+ 4\frac{16}{M^4} \sum_{i,j,k,l} \operatorname{cov}\{z_i d_j, z_k d_l\} \cos\left[\omega_a (t_i - t_j)\right] \cos\left[\omega_a (t_k - t_l)\right]
\end{aligned} \quad . \qquad (37)$$

Each of these three terms will now be determined separately.

4.1 Additive Noise Term

Considering that the additive is normally distributed with standard deviation σ_v, the covariance term in the second term of the second member of (37)

$$\operatorname{cov}\{d_i d_j, d_k d_l\} = \begin{cases} 2\sigma_v^4 & , i = j = k = l \\ \sigma_v^4 & , i = k \wedge j = l \wedge i \neq j \wedge k \neq l \\ \sigma_v^4 & , i = l \wedge j = k \wedge i \neq j \wedge k \neq l \\ 0 & , \text{otherwise} \end{cases} , \quad (38)$$

allows us to write

$$\begin{aligned} &\frac{16}{M^4} \sum_{i,j,k,l} \operatorname{cov}\{d_i d_j, d_k d_l\} \cos\left[\omega_s(t_i - t_j)\right] \cos\left[\omega_s(t_k - t_l)\right] = \\ &= \frac{16}{M^4} 2\sigma_v^4 M + 2\frac{16}{M^4}\sigma_v^4 \sum_{i \neq j} \cos^2\left[\omega_s(t_i - t_j)\right] = \\ &= \frac{16}{M^2} 2\sigma_v^4 M + 2\frac{16}{M^4}\sigma_v^4 \left(\frac{1}{2}M^2 - M\right) = \\ &= \frac{16}{M^2}\sigma_v^4 \end{aligned} . \quad (39)$$

4.2 Product of Signal by Additive Noise Term

Using the definition of covariance we can write, for the covariance between the products of z and d,

$$\begin{aligned} \operatorname{cov}\{z_i d_j, z_k d_l\} &= \mathrm{E}\{z_i d_j z_k d_l\} - \mathrm{E}\{z_i d_j\}\mathrm{E}\{z_k d_l\} = \\ &= \mathrm{E}\{z_i d_j z_k d_l\} \end{aligned} . \quad (40)$$

The terms $\mathrm{E}\{z_i d_j\}$ and $\mathrm{E}\{z_k d_l\}$ are null because z and d are independent and d has a null expected value. The remaining expected value in (40)

$$\mathrm{E}\{z_i d_j z_k d_l\} = \begin{cases} \mathrm{E}\{z_i d_j^2 z_k\} & , j = l \\ 0 & , j \neq l \end{cases} , \quad (41)$$

where

$$\mathrm{E}\{z_i d_j^2 z_k\} = \mathrm{E}\{z_i z_k\}\mathrm{E}\{d_j^2\} = \sigma_v^2 \mathrm{E}\{z_i z_k\} . \quad (42)$$

Equation (40)

$$\operatorname{cov}\{z_i d_j, z_k d_l\} = \begin{cases} \sigma_v^2 \mathrm{E}\{z_i z_k\} & , j = l \\ 0 & , j \neq l \end{cases} . \quad (43)$$

Inserting it into the quadruple summation of the last term in (37)

$$\begin{aligned} &\frac{16}{M^4} \sum_{i,j,k,l} \operatorname{cov}\{z_i d_j, z_k d_l\} \cos\left[\omega_a(t_i - t_j)\right] \cos\left[\omega_a(t_k - t_l)\right] = \\ &= \frac{16}{M^4} \sum_{i,j,k} \sigma_v^2 \mathrm{E}\{z_i z_k\} \cos\left[\omega_a(t_i - t_j)\right] \cos\left[\omega_a(t_k - t_j)\right] = \\ &= \frac{8}{M^3}\sigma_v^2 \sum_{i,k} \mathrm{E}\{z_i z_k\} \cos\left[\omega_a(t_i - t_k)\right] \end{aligned} . \quad (44)$$

Using the same reasoning as before leads to

$$\frac{16}{M^4}\sum_{i,j,k,l}\operatorname{cov}\left\{z_i d_j, z_k d_l\right\}\cos\left[\omega_a\left(t_i-t_j\right)\right]\cos\left[\omega_a\left(t_k-t_l\right)\right]= \\ =\frac{2}{M}\sigma_v^2 A^2 \quad . \qquad (45)$$

4.3 Signal Term

The value of sample voltage without noise, given by (4)

$$z_i = C + w_i \quad . \qquad (46)$$

with

$$w_i = A\cos\left(\omega_x t_i + \varphi\right) \quad . \qquad (47)$$

The covariance in the first term of the second member of (37)

$$\operatorname{cov}\left\{z_i z_j, z_k z_l\right\} = \operatorname{cov}\left\{\left(C+w_i\right)\left(C+w_j\right),\left(C+w_k\right)\left(C+w_l\right)\right\}= \\ =\operatorname{cov}\left\{C^2 + C\left(w_i+w_j\right)+w_i w_j, C^2 + C\left(w_k+w_l\right)+w_k w_l\right\} \quad . \qquad (48)$$

Using the properties of the covariance this can be written as

$$\operatorname{cov}\left\{z_i z_j, z_k z_l\right\}= \\ =4C^2\operatorname{cov}\left\{w_i, w_k\right\}+4C\times\operatorname{cov}\left\{w_i, w_k w_l\right\}+\operatorname{cov}\left\{w_i w_j, w_k w_l\right\} \quad . \qquad (49)$$

In the Appendix each of these covariances has been calculated. Inserting (67)(72)(77)(49)

$$\operatorname{cov}\left\{z_i z_j, z_k z_l\right\} = 2C^2 A^2 \cos\left(\omega_x t_i - \omega_x t_k\right) + \\ +\frac{1}{8}A^4\cos\left(\omega_x t_i + \omega_x t_j - \omega_x t_k - \omega_x t_l\right) \quad . \qquad (50)$$

Inserting this into the quadruple summation in the first term of the second member of (37)

$$\frac{16}{M^4}\sum_{i,j,k,l}\operatorname{cov}\left\{z_i z_j, z_k z_l\right\}\cos\left[\omega_x\left(t_i-t_j\right)\right]\cos\left[\omega_x\left(t_k-t_l\right)\right]=0 \quad . \qquad (51)$$

Inserting (39)(45)(51)(37)

$$\sigma^2_{A^2} = \frac{16}{M^2}\sigma_v^4 + \frac{8}{M}\sigma_v^2 A^2 \quad . \qquad (52)$$

5 MEAN OF ESTIMATED AMPLITUDE

We now know the mean and the variance of A^2 and are going to use it to compute the mean of A using the following approximation:

$$\mu_A \approx \sqrt{\mu_{A^2}} - \frac{\sigma^2_{A^2}}{8\sqrt{\mu^3_{A^2}}} \quad . \qquad (53)$$

This comes from a second order Taylor series of the non-linear function that is the square root.

Inserting (27)(52)(53)

$$\mu_A \approx \sqrt{A^2+\frac{4}{M}\sigma_v^2}-\frac{\frac{16}{M^2}\sigma_v^4+\frac{8}{M}\sigma_v^2 A^2}{8\left(A^2+\frac{4}{M}\sigma_v^2\right)^{\frac{3}{2}}}. \quad (54)$$

A simpler expression may be derived if we are not interested in the extreme cases of high random noise standard deviation and low number of samples, in which case some approximations can be made:

- The A^2 term in the denominator is usually much larger than the σ_v^2 term;
- The $\sigma_v^2 A^2$ term in the numerator is usually much larger than the σ_v^4 term.
- The square root can be substituted by a Taylor series approximation $\left(\sqrt{1+x}\approx 1+x/2\right)$.

The simplified expression than becomes

$$\mu_A \approx A+\frac{1}{M}\frac{\sigma_v^2}{A}. \quad (55)$$

This shows that the amplitude estimation is biased for a finite number of samples but asymptotically unbiased, that is, unbiased in the case of an infinite number of samples. It also shows that the expected value does not depend on the signal offset (C).

An interesting parameter to define is the relative error of the estimated amplitude:

$$\varepsilon_A \approx \frac{\mu_A - A}{A}. \quad (56)$$

Inserting (55)(56)

$$\varepsilon_A \approx \frac{1}{2M\cdot SNR^2}, \quad (57)$$

where

$$SNR=\frac{A}{\sqrt{2}\sigma_v}. \quad (58)$$

6 TRIANGULAR SIGNAL AMPLITUDE ESTIMATOR

All of the derivations done from Section II to Section V considered only the fundamental of the triangular signal. As stated before we intend to use the amplitude of the fundamental to estimate the amplitude of the triangular signal. Considering (2) we can write

$$A_{FUND}=\frac{8}{\pi^2}A_{TRI}, \quad (59)$$

where A_{FUND} is the amplitude of the fundamental of the triangular signal. Inserting (59) into (55) leads to

$$\mu_{A_{FUND}} \approx \frac{8}{\pi^2}A_{TRI}+\frac{1}{M}\frac{\sigma_v^2}{\frac{8}{\pi^2}A_{TRI}}. \quad (60)$$

So, we can define

$$\mu_{A_{TRI}} = \mu_{A_{FUND}} \frac{\pi^2}{8} = A_{TRI} + \frac{1}{M}\frac{\sigma_v^2}{A_{TRI}}\frac{\pi^4}{64}, \qquad (61)$$

where $\mu_{A_{TRI}}$ is the mean of the estimated amplitude of the triangular signal.

The relative error of the triangular wave estimation, given by

$$\varepsilon_{A_{TRI}} \approx \frac{\mu_{A_{TRI}} - A_{TRI}}{A_{TRI}}, \qquad (62)$$

7 MONTE CARLO ANALYSIS

In order to validate the approximations made, namely in (53)(55)

In Figure 2, the relative estimation error is plotted against the additive noise standard deviation. It is seen that, as expected, the error increases with the amount of noise. The vertical bars represent the confidence intervals in estimating the relative error using the Monte Carlo analysis with *J* repetitions. A 99.9 % confidence level was used to compute the intervals. It is assumed that the relative error estimation is normally distributed.

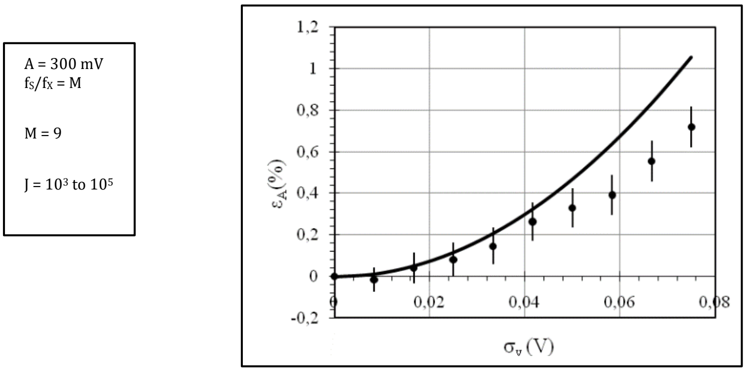

Figure 2: Expected value of the estimated triangular signal amplitude as a function of the additive noise standard deviation. The circles represent the values obtained with the Monte Carlo analysis. The confidence intervals for a confidence level of 99.9% are represented by the vertical bars. The solid line represents the theoretical value given by (62)

The parameters chosen for the test, and which are presented inside the box in the figure, were the ones that best illustrate the dependence of the relative error on the different parameters and which correspond to typical conditions encountered in practice.

8 EXPERIMENTAL VALIDATION

Whereas the previous section validated the mathematical approximations, this section reports experimental results that confirm the model itself represents conditions met in practice. The model assumes an ideal triangular signal of random initial phase corrupted by additive noise; all other non-ideal effects — jitter, phase noise, harmonic distortion, frequency error, quantisation, and so on — are deliberately excluded.

To gather the experimental data, a test setup like the one depicted in Figure 3 was built. A data acquisition board for a personal computer was used to sample and digitize the samples of a triangular signal created by a function generator to which normally distributed additive noise generated by the same generator was subtracted. This subtraction was achieved through the use of a differential input to the data acquisition board.

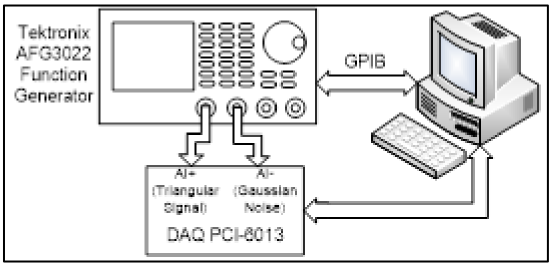

Figure 3: Test Bench. The personal computer controls the function generator using GPIB. The combination of the sinusoidal stimulus signal and the normally distributed noise is carried out inside the data acquisition board through the use of one of its differential inputs.

In Table 1 the pertinent characteristics of the instruments and the ADC used are presented.

Table 1: Manufacturer specifications for the instruments and ADC used.

Specification	**Value**
Function Generator (Tektronix AFG3022) [13]	
Triangle Wave Amplitude Accuracy	1 %
Triangle Wave Offset Accuracy	1% + 5 mV
Triangle Wave Frequency Accuracy	1 ppm+1 μHz
Total Harmonic Distortion	-70 dBc
Noise Bandwidth	25 MHz
ADC (National Instruments PCI-6023) [14]	
Number of Bits	16
Range	±5 V
Integral Non Linearity (INL)	1.5 LSB
Differential Non Linearity (DNL)	2 LSB
Gain Error	±75 ppm
Offset Error	372 μV
Bandwidth (small signal)	425 kHz
Quantization Step (Q)	0.152 mV
Noise standard deviation	0.9 LSB
Jitter	Not specified
Timing Accuracy (relative to sample rate)	100 ppm

One of the goals of the test setup is to be able to add a given amount of normally distributed noise to the signal that is sampled by the analog to digital converter. The function generator that produces the noise has a noise bandwidth of approximately 25 MHz (Table 1) while the data acquisition board has a small signal bandwidth of 425 kHz (this bandwidth is not, however, an equivalent noise bandwidth). Due to the bandwidths mismatch, in order to have a given value of noise standard deviation one has to produce an higher value of noise effective value that the desired one. To obtain this correction constant we estimated the standard deviation of noise produced as being equal to the root mean square (RMS) value of the residuals of sinefitting performed on a set of acquired data points affected by additive noise. Through trial and error the value of 118.4 for this correction constant was obtained. Fig. 4 shows the value of the estimated RMS error measured with the acquisition board as a function of desired noise standard deviation using the determined correction factor. Note that the amount of internal noise in the data acquisition board (in the order of μV) is negligible compared to the values of additive noise that were injected. The same can be said of the quantization error of the ADC.

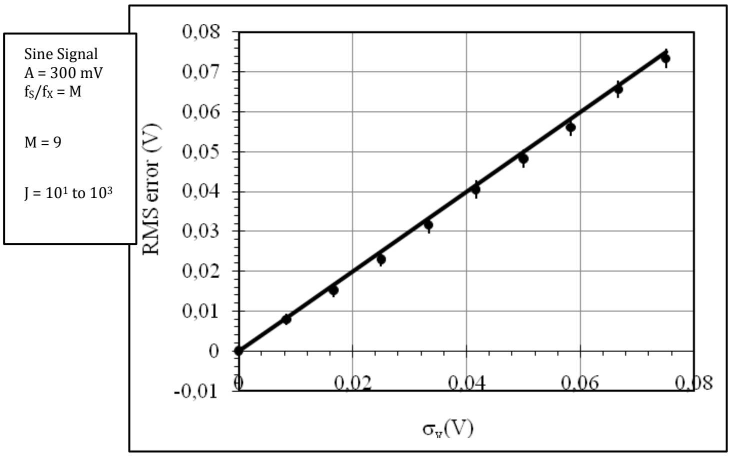

Figure 4: RMS error in function of the generated noise for a gain of 118.3952.

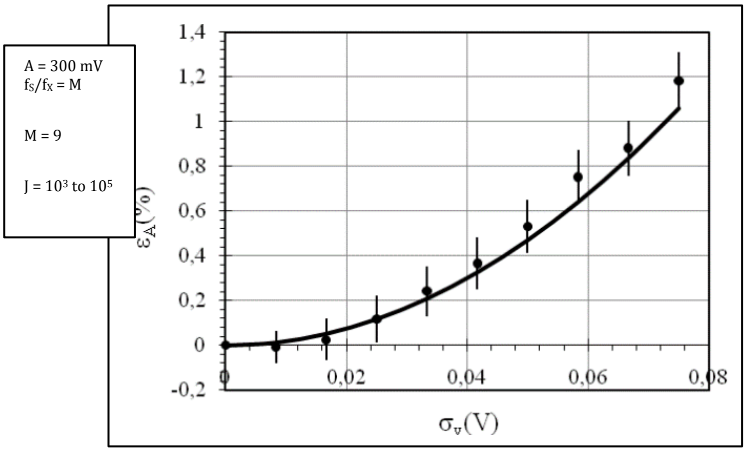

Figure 5: Experimentally determined relative error of the estimated amplitude as a function of the normalized additive noise standard deviation. The circles represent the values obtained with the Monte Carlo analysis done with a number of repetitions ranging from 10^3 (left most point) to 10^5 (right most point). The confidence intervals for a confidence level of 99.9% are represented by the vertical bars. The solid line represents the theoretical value given by (62). An offset was added to the experimental data so that the first data point had a value of 0 which is the theoretical value for the case of no additive noise present.

A Monte Carlo analysis was carried out to estimate the relative error of the sine fitting amplitude estimation as a function of additive noise standard deviation. The results are presented in **Error! Reference source not found.**. The vertical bars represent the confidence interval of the relative error estimation considering a normal distribution and a confidence level of 99.9%. The value of the relative error of the first data point was subtracted from all the data points. This was done because there was a discrepancy between the theoretical curve and the experimental points. This is most likely due to the error in the exact value of the amplitude of the triangular signal that was generated. The value of the relative error of the first point (left most) in Fig. 5 is thus 0 which is the value one expects when no noise is present. It can be seen that all the other data points are in agreement with the theoretical result since the confidence intervals all contain the solid line.

9 CONCLUSION

The experimental data presented does validate expression (62) proposed here to be used for the computation of the relative error of the estimation of the amplitude of a triangular signal using the three-parameter sine fitting algorithm to estimate the amplitude of its fundamental.

The expression derived is a simple one, due to some approximations done but which were thoroughly validated with simulated and experimental tests. It is, to the author's knowledge, a novel expression, and it captures the effects of both additive noise and a finite number of samples. In the future further work should be carried out to include other important effects like jitter, phase noise, quantization error and harmonic distortion. Note that expression (62) is an approximate expression which is not valid for the extreme cases of high random noise standard deviation (relative to signal amplitude) and low number of samples. In those cases, expression (54)

It should be pointed out that what the expression gives is the expected value of the error in the amplitude estimation and not the error itself which is random in nature. It thus cannot be used to correct the error but only to specify a confidence interval for the measurements made. It is also necessary, in the future, to demonstrate what the statistical distribution of the estimator is. Given the large number of samples generally used and the central limit theorem applied to (14)

The work presented here demonstrated that the estimator is asymptotically unbiased in the presence of additive noise.

Appendix

In this appendix, a simplified expression for

$$\begin{aligned}\mathrm{cov}\{z_i z_j, z_k z_l\} &= 4C^2\mathrm{cov}\{w_i, w_k\} + \\ &+ 4C\times\mathrm{cov}\{w_i, w_k w_l\} + \\ &+ Cov\{w_i w_j, w_k w_l\},\end{aligned} \quad (63)$$

is derived.

First term

The first covariance term in the second member can be written, using (47)

$$\mathrm{cov}\{w_i, w_k\} = \mathrm{cov}\{A\cos(\omega_x t_i + \varphi), A\cos(\omega_x t_k + \varphi)\}. \quad (64)$$

Using (32)(23)

$$\begin{aligned}\mathrm{cov}\{w_i, w_k\} &= A^2\mathrm{E}\{\cos(\omega_x t_i + \varphi)\cos(\omega_x t_k + \varphi)\} - \\ &- A^2\mathrm{E}\{\cos(\omega_x t_i + \varphi)\}\mathrm{E}\{\cos(\omega_x t_k + \varphi)\}\end{aligned}.$$

(65)

Transforming the product of cosines into a sum of cosines leads to

$$\begin{aligned}\mathrm{cov}\{w_i, w_k\} &= \frac{1}{2}A^2\mathrm{E}\{\cos(\omega_x t_i - \omega_x t_k)\} + \\ &+ \frac{1}{2}A^2\mathrm{E}\{\cos(\omega_x t_i + \omega_x t_k + 2\varphi)\} - \\ &- A^2\mathrm{E}\{\cos(\omega_x t_i + \varphi)\}\mathrm{E}\{\cos(\omega_x t_k + \varphi)\}\end{aligned}. \quad (66)$$

Since φ is a uniformly distributed random variable between 0 and 2π, $\mathrm{E}\{\cos(\omega_x t_i + \varphi)\}$ is null, which leads to

$$\mathrm{cov}\{w_i, w_k\} = \frac{1}{2}A^2\cos(\omega_x t_i - \omega_x t_k). \quad (67)$$

Second term

The second term in (63)(47)

$$\operatorname{cov}\{w_i, w_k w_l\} = \operatorname{cov}\begin{Bmatrix} A\cos(\omega_x t_i + \varphi), \\ A^2\cos(\omega_x t_k + \varphi)\cos(\omega_x t_l + \varphi) \end{Bmatrix}. \quad (68)$$

Transforming the product of cosines into a sum of cosines leads to

$$\operatorname{cov}\{w_i, w_k w_l\} = \frac{A^3}{2}\operatorname{cov}\begin{Bmatrix} \cos(\omega_x t_i + \varphi), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi) + \\ +\cos(\omega_x t_k - \omega_x t_l) \end{Bmatrix}. \quad (69)$$

Splitting the covariance of the sum into the sum of covariances leads to

$$\operatorname{cov}\{w_i, w_k w_l\} = \frac{1}{2}A^3\operatorname{cov}\begin{Bmatrix} \cos(\omega_x t_i + \varphi), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi) \end{Bmatrix} + \\ + \frac{1}{2}A^3\operatorname{cov}\begin{Bmatrix} \cos(\omega_x t_i + \varphi), \\ \cos(\omega_x t_k - \omega_x t_l) \end{Bmatrix}. \quad (70)$$

Substituting the covariances by expected values, as in (32)

$$\begin{aligned} \operatorname{cov}\{w_i, w_k w_l\} = {} & \frac{1}{2}A^3\mathrm{E}\{\cos(\omega_x t_i + \varphi)\cos(\omega_x t_k + \omega_x t_l + 2\varphi)\} \\ & - \frac{1}{2}A^3\mathrm{E}\{\cos(\omega_x t_i + \varphi)\}\mathrm{E}\{\cos(\omega_x t_k + \omega_x t_l + 2\varphi)\} + \\ & + \frac{1}{2}A^3\mathrm{E}\{\cos(\omega_x t_i + \varphi)\cos(\omega_x t_k - \omega_x t_l)\} - \\ & - \frac{1}{2}A^3\mathrm{E}\{\cos(\omega_x t_i + \varphi)\}\mathrm{E}\{\cos(\omega_x t_k - \omega_x t_l)\} \end{aligned} \quad (71)$$

Since φ is a uniformly distributed random variable between 0 and 2π, $\mathrm{E}\{\cos(\omega_x t_i + \varphi)\}$ is null, which leads to

$$\operatorname{cov}\{w_i, w_k w_l\} = 0. \quad (72)$$

Third term

Again inserting (47)(63)

$$\operatorname{cov}\{w_i w_j, w_k w_l\} = \operatorname{cov}\begin{Bmatrix} A^2\cos(\omega_x t_i + \varphi)\cos(\omega_x t_j + \varphi), \\ A^2\cos(\omega_x t_k + \varphi)\cos(\omega_x t_l + \varphi) \end{Bmatrix}. \quad (73)$$

Transforming the product of cosines into a sum of cosines leads to

$$\operatorname{cov}\{w_i w_j, w_k w_l\} = \frac{1}{4}A^4\operatorname{cov}\begin{Bmatrix} \cos(\omega_x t_i + \omega_x t_j + 2\varphi) + \\ +\cos(\omega_x t_i - \omega_x t_j), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi) + \\ +\cos(\omega_x t_k - \omega_x t_l) \end{Bmatrix}. \quad (74)$$

Substituting the covariance of the sum into the sum of covariances leads to

$$\operatorname{cov}\{w_i w_j, w_k w_l\} = \frac{1}{4}A^4 \operatorname{cov}\left\{\begin{matrix}\cos(\omega_x t_i + \omega_x t_j + 2\varphi), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi)\end{matrix}\right\} + \\ + \frac{1}{4}A^4 \operatorname{cov}\left\{\begin{matrix}\cos(\omega_x t_i + \omega_x t_j + 2\varphi), \\ \cos(\omega_x t_k - \omega_x t_l)\end{matrix}\right\} + \\ + \frac{1}{4}A^4 \operatorname{cov}\left\{\begin{matrix}\cos(\omega_x t_i - \omega_x t_j), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi)\end{matrix}\right\} + \\ + \frac{1}{4}A^4 \operatorname{cov}\left\{\begin{matrix}\cos(\omega_x t_i - \omega_x t_j), \\ \cos(\omega_x t_k - \omega_x t_l)\end{matrix}\right\} . \quad (75)$$

Since φ is a uniformly distributed random variable between 0 and 2π, the 2nd and 3rd terms are null. Furthermore, since the argument of the covariance in the last term is not a random variable, this term is also null. What remains is thus

$$\operatorname{cov}\{w_i w_j, w_k w_l\} = \frac{1}{4}A^4 \operatorname{cov}\left\{\begin{matrix}\cos(\omega_x t_i + \omega_x t_j + 2\varphi), \\ \cos(\omega_x t_k + \omega_x t_l + 2\varphi)\end{matrix}\right\} . \quad (76)$$

This covariance finally leads to

$$\operatorname{cov}\{w_i w_j, w_k w_l\} = \frac{1}{8}A^4 \cos(\omega_x t_i + \omega_x t_j - \omega_x t_k - \omega_x t_l) . \quad (77)$$

REFERENCES

[1] B.S. Gaudi, K.Z. Stanek, J.D. Hartman, M.J. Holman, B.A. McLeod, “On the rotation period of (90377) Sedna”, Astrophysical Journal 629 (2005) L49–L52.

[2] N. Pears, B.J. Liang, Z.Z. Chen, “Mobile Robot Visual Navigation Using Multiple Features”, EURASIP Journal on Applied Signal Processing VL (2005) 2250–2259.

[3] “IEEE Standard for Digitizing Waveform Recorders,” IEEE Std. 1057-2007, April 2008.

[4] “IEEE Standard for Terminology and Test Methods for Analog-to-Digital Converters”, IEEE Std 1241-2010 (Revision of IEEE Std 1241-2000), January 2011. https://air.unipr.it/handle/11381/2303408, https://doi.org/10.1109/IEEESTD.2011.5692956.

[5] Pedro M. Ramos, Manuel Fonseca da Silva, António Manuel da Cruz Serra, “Low Frequency Impedance Measurement Using Sine-Fitting”, Measurement, vol. 35, no. 1, pp. 89-96, January 2004.

[6] A. Moschitta and P. Carbone, “Cramér–Rao Lower Bound for Parametric Estimation of Quantized Sinewaves”, IEEE Transactions on Instrumentation and Measurement, vol. 56, no. 3, pp. 975-982, June 2007.

[7] Tomas Andersson, Peter Händle, “IEEE Standard 1057, Cramér-Rao Bound and the Parsimony Principle”, IEEE Transactions on Instrumentation and Measurement, vol. 55, no. 1, pp. 44-53, February 2006.

[8] Tomas Andersson, Peter Händle, “IEEE Standard 1057, Cramér-Rao Bound and the Parsimony Principle”, Proc. Of 8th International Workshop on ADC Modelling and Testing IWADC2003, September 8-10, 2003, pp. 231-234.

[9] Tomas Andersson, “Selected topics in frequency estimation,” Licentiate Thesis TRITA–S3–SB–0323, Royal Institute of Technology, May 2003. Available at www.s3.kth.se.

[10] Peter Händle, “Properties of the IEEE-STD-1057 Four-Parameter Sine Wave Fit Algorithm”, IEEE Transactions on Instrumentation and Measurement, vol. 49, no. 6, pp. 1189-1193, December 2000.

[11] Frans Verbeyst, Yves Rolain, Rik Pintelon, Johan Schoukens, “Enhanced Time Base Jitter Compensation of Sine Waves”, Proc. Instrumentation and Measurement Technology Conference, pp. 1-5, 1-3 May 2007.

[12] T. Michael Souders, Donald R. Flach, Charles Hagwood and Grace L. Yang, “The Effects of Timing Jitter in Sampling Systems”, IEEE Transactions on Instrumentation and Measurement, vol. 39, no. 1, pp. 80-85, February 1990.

[13] "Arbitrary/Function Generators AFG 3011 / 3021B / 3022B / 3101 / 3102 / 3251 / 3252 Data Sheet", Tektronix, July 2009. Available at http://www2.tek.com/cmsreplive/psrep/13567/76W_18656_3_2009.07.16.16.20.18_13567_EN.pdf

[14] "NI 6023E/6024E/6025E Family Specifications", National Instruments, December 2005. Available at http://www.ni.com/pdf/manuals/ 370719c.pdf.

[15] F. Corrêa Alegria, A. Cruz Serra, "Uncertainty of the estimates of sine wave fitting of digital data in the presence of additive noise", Proc. 2006 IEEE Instrumentation and Measurement Technology Conference (IMTC), Sorrento, Italy, April 2006.https://doi.org/10.1109/IMTC.2006.328187

[16] F. Corrêa Alegria, A. Cruz Serra, "Standard histogram test precision of ADC gain and offset error estimation", IEEE Transactions on Instrumentation and Measurement, vol. 56, no. 5, pp. 1527–1531, 2007.https://doi.org/10.1109/TIM.2007.907978

[17] F. Corrêa Alegria, A. M. da Cruz Serra, "Uncertainty of ADC random noise estimates obtained with the IEEE 1057 standard test", IEEE Transactions on Instrumentation and Measurement, vol. 54, no. 1, pp. 110–116, 2005.https://doi.org/10.1109/TIM.2004.840226

[18] F. Corrêa Alegria, P. Arpaia, P. Daponte, A. Cruz Serra, "ADC histogram test using small-amplitude input waves", Proc. XVI IMEKO World Congress, Vienna, Austria, September 2000.

[19] F. Corrêa Alegria, E. Martinho, F. Almeida, "Measuring soil contamination with the time domain induced polarization method using LabVIEW", Measurement, vol. 42, no. 7, pp. 1082–1091, 2009.https://doi.org/10.1016/j.measurement.2009.03.015

[20] S. Guillén-Ludeña, M. J. Franca, F. Alegria, A. J. Schleiss, A. H. Cardoso, "Hydromorphodynamic effects of the width ratio and local tributary widening on discordant confluences", Geomorphology, vol. 293, pp. 289–304, 2017.https://doi.org/10.1016/j.geomorph.2017.06.006

[21] O. Birjukova, S. Guillén-Ludeña, F. Alegria, A. H. Cardoso, "Three-dimensional flow field at confluent fixed-bed open channels", Proc. River Flow 2014, pp. 1007–1014, 2014.https://doi.org/10.1201/b17133-136